An introduction to topological K-Theory and Snaith's Theorem

Robin Stoll

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Betreuer: Dr. Urs Schreiber

Zweitgutachter: Dr. Viktoriya Ozornova

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Zusammenfassung

K-Theorie umfasst viele verschiedene Fachbereiche, darunter Topologie, Algebra, Algebraische Geometrie, Analysis und Physik. Der erste Teil dieser Arbeit, Kapitel 2, konzentriert sich darauf, die Grundlagen der topologischen Sicht einzuführen. Die wichtigsten Objekte, die wir betrachten, werden mit Hilfe von zwei unterschiedlichen Äquivalenzrelationen auf der Menge der Vektorbündel über einem Raum konstruiert. Sie korrespondieren zu zwei verschiedenen Varianten der *Stabilisierung*, d.h. dem Identifizieren von Bündeln, die sich nur um triviale Bündel unterscheiden.

Wir übertragen die algebraische Struktur der direkten Summe und des Tensorprodukts von Vektorbündeln auf unsere Mengen. Dadurch werden sie eine Ringstruktur bekommen. Weiterhin können wir daraus eine Kohomologietheorie konstruieren.

Das sogenannte Bott Periodizitäts Theorem (dessen Beweis wir skizzieren werden), eines der Hauptresultate über topologische K-Theorie, wird uns sagen, dass diese Kohomologietheorie zweiperiodisch ist. Schließlich, um zu sehen, dass unsere gerade entwickelte Theorie tatsächlich nützlich ist, werden wir einige ihrer Hauptanwendungen nennen: Die Lösung des Hopfinvariante-1-Problems, sowie Klassifizierunsgsresultate über Divionsalgebren und Parallelisierbarkeit von Sphären.

Im zweiten Teil dieser Arbeit, bestehend aus Kapitel 3 und 4, präsentieren wir eine alternative Methode topologische K-Theorie zu konstruieren, die auf $\mathbb{C}P^\infty$ basiert. Da dieser Raum Linienbündel klassifiziert, werden wir, in einem gewissen Sinne, die Objekte, die wir im ersten Teil aus allen Vektorbündeln konstruiert haben aus nur den eindimensionalen erhalten. Um dies zu tun, werden wir einige der Konzepte aus stabiler Homotopietheorie benötigen. Wir werden hauptsächlich Spektren und ihre assozierten (Ko-)Homologietheorien benötigen, aber es werden auch einige andere verwandte Dinge eingeführt werden.

Introduction

K-theory encompasses a broad range of fields, among them topology, algebra, algebraic geometry, analysis and physics. The first main part of this thesis, Chapter 2, focuses on introducing the foundations of the topological view on it. The main objects of our study will be constructed from the set of vector bundles over a space by considering two equivalence relations on it. They correspond to two different versions of *stabilizing*, i.e. identifying vector bundles which only differ by trivial bundles.

We will transfer the algebraic structure on vector bundles given by direct sum and tensor product to these sets. This will make our objects into rings. Moreover we can construct a cohomology theory from them. The so called Bott periodicity theorem (the proof of which we sketch), one of the main results about topological K-theory, tells us that this cohomology theory is two periodic. Finally, to see that the theory we developed is actually useful, we state some of the classical main applications: the solution to the Hopf Invariant One Problem, as well as classification results about division algebras over $\mathbb{R}$ and parallelizability of spheres.

In the second part of this thesis, consisting of Chapters 3 and 4, we present an alternative way of constructing topological K-theory which is based on $\mathbb{C}P^\infty$. Hence, since this space classifies line bundles, we will in some sense obtain the objects we constructed in the first part using all vector bundles from only the one dimensional ones. To do this we will need quite a bit of the concepts and language from stable homotopy theory. Mainly we will introduce spectra and their associated (co)homology theories, but there we will also be some related things sprinkled in.

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Contents

1 Preliminaries

1.1 Notation

Definition 1.1. We write lim for categorical limits and colim for categorical colimits.

Definition 1.2. We denote by **Top** the category of topological spaces with continuous maps and by $\mathbf{Top}_*$ the category of pointed topological spaces with pointed continuous maps.

We also write **CW** for the category of CW-complexes with continuous maps and $\mathbf{CW}_*$ for the category of pointed CW-complexes (for us this will mean pointed in a 0-cell) with pointed continuous maps.

Furthermore, we denote by $\mathbf{hCW}_*$ the category of pointed CW-complexes with morphisms the equivalence classes of pointed continuous maps up to homotopy relative to the basepoint and by **hCW** the non-pointed analog.

When we talk about a map between topological spaces, it is always meant to lie in the appropriate category. In particular it will be continuous.

If we introduce a pointed space X, the notation x_0 for its basepoint will be implicit. The corresponding analog also holds for Y with y_0 and for other letters.

Definition 1.3. For X and Y two pointed spaces, we will write $[X, Y]$ for the set of pointed maps $X \to Y$ up to basepoint preserving homotopy.

Definition 1.4. For a space X, we write X_+ for the pointed space $(X \amalg \{*\}, *)$, i.e. we adjoin a disjoint basepoint to X. This gives us, in a canonical way, functors $(-)_+ \colon \mathbf{Top} \to \mathbf{Top}_*$ and $(-)_+ \colon \mathbf{CW} \to \mathbf{CW}_*$.

Definition 1.5. Let X be a space. We denote by $\mathrm{S}X$ the *(unreduced) suspension* of X, i.e. the quotient of $X \times [0, 1]$ by the equivalence relation identifying both $X \times \{0\}$ and $X \times \{1\}$ to a point each.

Definition 1.6. Let X and Y be two pointed spaces. We denote by $X \vee Y$ the *wedge sum*, i.e. the subspace $X \times y_0 \cup x_0 \times Y$ of $X \times Y$ with basepoint (x_0, y_0), and by $X \wedge Y$ the *smash product*, i.e. the quotient $(X \times Y)/(X \vee Y)$ with basepoint $[(x_0, y_0)]$. These constructions are functorial in both variables.

We also write $\Sigma X = \mathrm{S}^1 \wedge X$, the *reduced suspension* of X.

We could also have defined ΣX to be the quotient of $\mathrm{S}X$ by (the image of) $\{x_0\} \times [0, 1]$. Hence the "reduced" in its name. Also note that $\Sigma \mathrm{S}^n \cong \mathrm{S}^{n+1}$.

Remark 1.7. If X and Y are CW-complexes, we want $X \wedge Y = (X \times Y)/(X \vee Y)$ to also be a CW-complex. For this to be the case, we have to give $X \times Y$ the weak topology. When talking about the smash product of two CW-complexes, we will always mean it in this sense. Luckily, if either X or Y has finitely many cells or both have only countably many, the topologies agree (in particular for ΣX), so one does not have to be too careful in most cases. For more details see [HaAT, Chapter 0 and Appendix].

For X and Y two CW-complexes, the smash product $X \wedge Y$ consists of the basepoint (a 0-cell) and a $(n + m)$-cell for each pair (C_X^n, C_Y^m) where C_X^n is a non-basepoint n-cell of X and C_Y^m is a non-basepoint m-cell of Y.

Definition 1.8. For X a pointed space, we define the *reduced cone* of X to be the space $\widetilde{\mathrm{C}}X = (X \times \mathrm{I})/(X \times \{0\} \cup \{x_0\} \times \mathrm{I})$ with basepoint $[(x_0, 0)]$.

Also, given two pointed spaces X and Y and a pointed map $f\colon X \to Y$, we set the *reduced mapping cone* of f to be the pointed space $Y \cup_f \widetilde{\mathrm{C}}X$, i.e. the reduced cone $\widetilde{\mathrm{C}}X$ glued to Y along f.

By definition there is a homeomorphism $\widetilde{\mathrm{C}}X \cong X \wedge (\mathrm{I}, 0)$.

Definition 1.9. A space X is *normal*, or T_4, if it is T_1, i.e. points are closed, and for each pair of disjoint closed subsets $A_1, A_2 \subseteq X$, there exist disjoint open subsets $U_1, U_2 \subseteq X$ such that $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$.

Definition 1.10. By a *vector bundle* we will mean a complex vector bundle, though without requiring the dimension of the fibers to be constant. Since we still have local trivializations $p^{-1}(U) \to U \times \mathbb{C}^n$, the dimension of the fibers must still be locally constant, i.e. constant on each connected component.

We denote by $\mathrm{Vect}(X)$ the set of vector bundles over X up to isomorphism. We write $\underline{\mathbb{C}}^n \in \mathrm{Vect}(X)$ for the n-dimensional trivial bundle over X, omitting the base space in the notation.

If $E \in \mathrm{Vect}(X)$ is a vector bundle and $A \subseteq X$, we write $E|_A$ for the vector bundle restricted to A. Similarly, for $x \in X$, we denote by E_x the fiber of E over x.

1.2 Point-set lemmas

We now collect some statements about general topology we will need later.

1.2.1 about homotopy equivalences

Definition 1.11. A pointed space X is called *well-pointed* if the pair $(X, \{x_0\})$ has the homotopy extension property.

Lemma 1.12. *Let X be a well-pointed space. Then the quotient map $\mathrm{S}X \to \Sigma X$ from the unreduced to the reduced suspension is a homotopy equivalence.*

A proof can be found in [Bre, VII, Theorem 1.9].

Lemma 1.13. *Let X and Y be two spaces, $f\colon X \to Y$ a homotopy equivalence, and $x_0 \in X$, $y_0 \in Y$ two basepoints such that (X, x_0) and (Y, y_0) are well-pointed. If $f(x_0) = y_0$, then f is also a homotopy equivalence relative to the basepoints.*

This is a special case of [HaAT, Proposition 0.19].

Lemma 1.14. *Let X and Y be two homotopy equivalent, path-connected spaces. Also let $x_0 \in X$ and $y_0 \in Y$ such that (X, x_0) and (Y, y_0) are well-pointed. Then (X, x_0) and (Y, y_0) are also homotopy equivalent relative to the basepoints.*

Proof. Choose a homotopy equivalence $f\colon X \to Y$. If $f(x_0) = y_0$, we are done by the previous lemma. Otherwise we can replace f by a homotopic map f' with $f'(x_0) = y_0$. To construct f', choose a path γ from $f(x_0)$ to y_0. Now define

$F\colon X \times \{0\} \cup \{x_0\} \times [0,1]$ by setting it to be f on X and γ on $\{x_0\} \times [0,1]$. Since (X, x_0) is well-pointed (i.e. has the homotopy extension property), we can extend F to a map G on all of $X \times [0,1]$. Now $f' = G|_{X\times\{1\}}$ has the required property. □

1.2.2 about compact Hausdorff spaces

Lemma 1.15. *If X is a compact Hausdorff space and $A \subseteq X$ is closed, then X/A is again compact and Hausdorff.*

Proof. It is compact, since it is the image of the compact space X under the continuous map $q\colon X \to X/A$.

We show Hausdorff directly from the definition, i.e. that there exists a pair of disjoint open neighborhoods U_1, U_2 for every pair of different points $q(x_1), q(x_2) \in X/A$ (here $x_1, x_2 \in X$).

If both $q(x_1), q(x_2) \in (X/A) \setminus (A/A)$, we can take the disjoint open neighborhoods $U_1 = q(V_1 \cap (X \setminus A))$ and $U_2 = q(V_2 \cap (X \setminus A))$ where V_1 and V_2 are disjoint open neighborhoods of x_1 and x_2 respectively (in X). The subspaces U_1 and U_2 are open by the definition of the quotient topology.

In the other case, i.e. without loss of generality $q(x_1) = A/A$, we use that, since X is compact and Hausdorff and thus normal, we have disjoint open sets V_1, V_2 containing A and x_2 respectively. Their images $q(V_1)$ and $q(V_2)$ are again disjoint and open by the quotient topology. □

Lemma 1.16. *If X and Y are pointed compact Hausdorff spaces, then $X \vee Y$ and $X \wedge Y$ are compact and Hausdorff as well. In particular ΣX and $\widetilde{\mathrm{C}}X$ are compact and Hausdorff.*

Proof. The space $X \vee Y$ is compact since it is the image of the compact space $X \amalg Y$ under the map identifying the two basepoints. As a subspace of the Hausdorff space $X \times Y$, it is also Hausdorff.

Since $X \vee Y$ is compact and $X \times Y$ is Hausdorff, it is a closed subset. Hence, by Lemma 1.15, $X \wedge Y$ is compact and Hausdorff as well. □

Lemma 1.17. *Let X and Y be two pointed compact Hausdorff spaces. There is a natural homeomorphism*

$$\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$$

Proof. The inclusions $X \to X \vee Y$ and $Y \to X \vee Y$ induce maps $\Sigma X \to \Sigma(X \vee Y)$ and $\Sigma Y \to \Sigma(X \vee Y)$ and hence a map $\Sigma X \vee \Sigma Y \to \Sigma(X \vee Y)$. This is bijective and a homeomorphism, since, by the last lemma, domain and target are compact and Hausdorff. The naturality follows from the naturality of the inclusions of X respectively Y into $X \vee Y$. □

Lemma 1.18. *Let X is a pointed compact Hausdorff space and $A \subseteq X$ a pointed inclusion with A closed. Then the reduced suspension commutes with the quotient, i.e.*

$$\Sigma X / \Sigma A \cong \Sigma(X/A)$$

and this is natural with respect to maps of such pairs.

Proof. Let $q\colon X \to X/A$ be the quotient map. We get a factorization

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Sigma q} & \Sigma(X/A) \\ & \searrow \quad \nearrow & \\ & \Sigma X/\Sigma A & \end{array}$$

which is bijective. Since, by Lemma 1.15 and Lemma 1.16, $\Sigma X/\Sigma A$ as well as $\Sigma(X/A)$ are compact and Hausdorff, the map is already a homeomorphism. The naturality follows directly from the definitions of the involved maps. □

1.2.3 about the one-point compactification and smash products

Definition 1.19. Let X be a space. By X^* we will denote the one-point compactification of X, i.e. the set $X \amalg \{\infty\}$ with the topology

$$\{U \mid U \subseteq X \text{ open}\} \cup \{(X \setminus K) \cup \{\infty\} \mid K \subseteq X \text{ compact and closed}\}$$

When we use X^* as a pointed space, it has the canonical basepoint ∞.

Lemma 1.20. *Let X be a compact Hausdorff space and $x \in X$. Then the map*

$$f\colon (X \setminus \{x\})^* \longrightarrow X, \quad p \mapsto \begin{cases} p & p \neq \infty \\ x & p = \infty \end{cases}$$

is a homeomorphism.

Proof. Since $(X \setminus \{x\})^*$ is compact, X is Hausdorff, and f is bijective, it is enough to show that it is continuous. The preimage of an open set not containing x is again open. If $x \in U \subseteq X$ is open, its complement $X \setminus U$ is closed and hence compact. Thus $f^{-1}(U) = (U \setminus \{x\}) \cup \{\infty\}$ is open in $(X \setminus \{x\})^*$. □

Lemma 1.21. *Let X and Y be two locally compact (i.e. each point has a closed, compact neighborhood) Hausdorff spaces. Then the map*

$$f\colon (X \times Y)^* \longrightarrow X^* \wedge Y^*$$

given by $(x, y) \mapsto [(x, y)]$ and $\infty \mapsto [(\infty, \infty)]$ is a homeomorphism.

Proof. The composition

$$X \times Y \xrightarrow{\iota} X^* \times Y^* \xrightarrow{q} X^* \wedge Y^*$$

of the inclusion ι and the quotient map q is an embedding. Hence we can consider $X \times Y$ to be a subspace of $X^* \wedge Y^*$. In particular we have $X \times Y = (X^* \wedge Y^*) \setminus \{[(\infty, \infty)]\}$.

Since X and Y are locally compact and Hausdorff, the one-point compactifications X^* and Y^* are Hausdorff as well. Hence, by Lemma 1.16, $X^* \wedge Y^*$ is compact and Hausdorff.

We now apply the previous lemma to obtain the desired result. □

From this lemma we directly get the following well-known result.

Corollary 1.22. *There is a homeomorphism* $\mathrm{S}^n \wedge \mathrm{S}^m = \mathrm{S}^{n+m}$.

1.3 Extending trivializations

In this subsection, we will prove the following lemma, which we will need later.

Lemma 1.23. *Let X be a compact Hausdorff space, $A \subseteq X$ closed and E a vector bundle over X which is trivial when restricted to A. If $h\colon A \times \mathbb{C}^n \to E|_A$ is a trivialization, then there is an open neighborhood U of A and a trivialization $h'\colon U \times \mathbb{C}^n \to E|_U$ such that $h'|_{A\times\mathbb{C}^n} = h$.*

Before we can proof this lemma, we need the following statements.

Theorem 1.24 (Tietze extension theorem)**.** *If X is normal, $A \subseteq X$ closed and $f\colon A \to \mathbb{R}$ a map, then there exists a map $g\colon X \to \mathbb{R}$ such that $g|_A = f$.*

This can be found in [Bre, I, Theorem 10.4].

Corollary 1.25. *If X is normal, $A \subseteq X$ closed and $f\colon A \to \mathbb{C}^n$, then there exists a map $g\colon X \to \mathbb{C}^n$, such that $g(a) = f(a)$ for all $a \in A$.*

Proof. Since we have $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we can consider f to be $2n$ maps $f_i\colon A \to \mathbb{R}$. Each of these can be extended to a function g_i on all of X. Together they give us a map $g\colon X \to \mathbb{R}^{2n}$ which extends f. □

Lemma 1.26. *If X is normal and $U \subseteq X$ an open neighborhood of a point $x \in X$, we can find a closed (in X) neighborhood A of x with $A \subseteq U$.*

Proof. By normality of X, we can find disjoint opens $U_1, U_2 \subseteq X$ with $x \in U_1$ and $X \setminus U \subseteq U_2$. Then $A = X \setminus U_2 \subseteq U$ is closed in X and contains the neighborhood $V = U_1$ of x. □

We are now ready to prove the lemma.

Proof of Lemma 1.23. First we note again that compact Hausdorff spaces are normal, hence we can apply the two previous results.

Consider a family of open subsets $(U_a)_{a\in A}$, with $a \in U_a \subseteq X$, such that E is trivial over each U_a. Now let $B_a \subseteq U_a$ be a closed neighborhood of a (which exists by the previous lemma) and $h_a\colon B_a \times \mathbb{C}^n \to E|_{B_a}$ trivializations of E.

Since E is trivial over A, there exist $n = \dim E|_A$ linearly independent sections $s_i\colon A \to E|_A$, $i = 1, \dots, n$. Consider the maps $\pi_{\mathbb{C}^n} \circ h_a^{-1} \circ (s_i|_{A\cap B_a})\colon A \cap B_a \to \mathbb{C}^n$, where $\pi_{\mathbb{C}^n}$ is the projection $B_a \times \mathbb{C}^n \to \mathbb{C}^n$. By Corollary 1.25 and the fact that closed subsets of normal spaces are normal, these can be extended to $t'_{ia}\colon B_a \to \mathbb{C}^n$. From these we get sections $t_{ia}\colon B_a \to E|_{B_a}$ given by $b \mapsto h_a(b, t'_{ja}(b))$.

Since B_a is a neighborhood of a, there exists $V_a \subseteq X$ open such that $a \in V_a \subseteq B_a$. The family $(V_a)_{a\in A}$ is an open cover of A. Because A is a closed subset of a compact space and hence compact, there exists a finite subset $J \subseteq A$ such that

$(V_j)_{j\in J}$ covers A. Since X is compact, thus paracompact, and also Hausdorff, there is a partition of unity $\phi_j, \phi\colon X \to [0,1]$ subordinate to the open cover $\{V_j, X\setminus A\}$ of X. Let $V = \bigcup_{j\in J} V_j$.

We now define sections $t_i\colon V \to E|_V$ by $t_i(v) = \sum_{j\in J} \phi_j t_{ij}$. These extend the s_i. Let $d\colon V \to \mathbb{C}$ be defined by $d(v) = \det(t_1(v), \ldots, t_n(v))$. Since the s_i are linearly independent, d is nonzero on A and thus on an open (in V and thus X) neighborhood U of A. Hence the t_i are linearly independent on U. This gives us a trivialization h' extending h onto U. □

2 An introduction to topological K-Theory

The idea of topological K-theory is to obtain algebraic invariants for spaces from the topological data given by the set of vector bundles over them. In this section we will consider two notions of equivalence for vector bundles, provide the corresponding sets of equivalence classes with the structure of rings, and show that there is a strong relation between these two objects. Naturally these invariants will be functorial with respect to continuous maps. Furthermore we will state (and very roughly sketch a proof of) Bott periodicity, one of the most fundamental theorems in K-theory. Using this, we obtain a cohomological structure from these rings. We will mostly follow the (unfinished) book by Hatcher [HaK], though many things will be presented in more detail.

In this section all spaces are assumed to be compact and Hausdorff.

2.1 The group $\widetilde{\mathrm{K}}(X)$

We now introduce the first, and most important, of the objects we are going to study.

Definition 2.1. Let E_1 and E_2 be two vector bundles over X. We call them *reduced stably isomorphic*, written $E_1 \sim E_2$, if there exist $n_1, n_2 \in \mathbb{N}_0$ such that $E_1 \oplus \underline{\mathbb{C}}^{n_1} \cong E_2 \oplus \underline{\mathbb{C}}^{n_2}$. This defines an equivalence relation.

By $\widetilde{\mathrm{K}}(X)$ we denote the set of $\sim$-equivalence classes of vector bundles over a nonempty space X.

As always in algebraic topology, we want to equip $\widetilde{\mathrm{K}}(X)$ with more structure. In the next lemmas we will provide it the structure of an abelian group, in a later subsection we will then make it into a (non-unital) ring.

Lemma 2.2. *The operation $\oplus$ descends to a well defined operation on $\widetilde{\mathrm{K}}(X)$ which we denote by $+$. It is commutative, associative and has a neutral element given by the equivalence class of $\underline{\mathbb{C}}^0$.*

Proof. We need to show that if $E_1 \sim E_1'$ and $E_2 \sim E_2'$, i.e. $E_1 \oplus \underline{\mathbb{C}}^n \cong E_1' \oplus \underline{\mathbb{C}}^{n'}$ and $E_2 \oplus \underline{\mathbb{C}}^m \cong E_2' \oplus \underline{\mathbb{C}}^{m'}$, then $E_1 \oplus E_2 \sim E_1' \oplus E_2'$. This implies $E_1 \oplus \underline{\mathbb{C}}^n \oplus E_2 \oplus \underline{\mathbb{C}}^m \cong E_1' \oplus \underline{\mathbb{C}}^{n'} \oplus E_2' \oplus \underline{\mathbb{C}}^{m'}$ and hence $E_1 \oplus E_2 \oplus \underline{\mathbb{C}}^{n+m} \cong E_1' \oplus E_2' \oplus \underline{\mathbb{C}}^{n'+m'}$ which gives the needed conclusion.

The other statements now follow immediately from the corresponding facts for $\oplus$. ☐

Lemma 2.3. *For every vector bundle E over X, there exists a vector bundle E' over X and $n \in \mathbb{N}_0$ such that $E \oplus E' \cong \underline{\mathbb{C}}^n$.*

Proof. For a proof of the statement for the usual definition of vector bundles (with constant fiber dimension), see [HaK, Proposition 1.4].

Let $m = \max_{x \in X} \dim E_x$. This is finite since X is compact. Now let E'' be the vector bundle which is trivial over each connected component with dimension m minus the dimension of E over that connected component. Then $E \oplus E''$ is a vector bundle in the usual sense and we can apply the version of the lemma which we cited above. ☐

This lemma is one of the main reasons we want our spaces to be Hausdorff and compact. The statement would be false if one of these conditions were omitted.

From the last two lemmas follows immediately:

Proposition 2.4. *The set $\widetilde{\mathrm{K}}(X)$, together with the operation $+$, forms an abelian group. We will call it the* reduced K-group *of X.*

We can already compute the following two, very easy, examples.

Lemma 2.5. *We have $\widetilde{\mathrm{K}}(\mathrm{pt}) \cong 0$ and $\widetilde{\mathrm{K}}(\mathrm{S}^1) \cong 0$.*

Proof. All vector bundles over a point are trivial and thus reduced stably isomorphic to $[\underline{\mathbb{C}}^0]$.

Let D^1_+ respectively D^1_- be the upper respectively lower semicircle (to be precise, we take them to be open and a little extended). Now any n-dimensional vector bundle $E \to \mathrm{S}^1$ is trivial over both D^1_+ and D^1_- since they are contractible. Hence E is completely determined by a clutching function $f \colon \mathrm{S}^0 \to \mathrm{GL}_n(\mathbb{C})$ (since $\mathrm{D}^1_+ \cap \mathrm{D}^1_- \simeq \mathrm{S}^0$). But $\mathrm{GL}_n(\mathbb{C})$ is path-connected. Thus any such f is homotopic to a constant map and hence $E \cong \underline{\mathbb{C}}^n$. ☐

2.2 The group $\mathrm{K}(X)$

We now want to consider a slightly different notion of equivalence on vector bundles over a space. In contrast to the last one it remembers the dimension of the vector bundles.

Definition 2.6. Let E_1 and E_2 be two vector bundles over X. We call them *stably isomorphic*, written $E_1 \cong_s E_2$, if there exists $n \in \mathbb{N}_0$ such that $E_1 \oplus \underline{\mathbb{C}}^n \cong E_2 \oplus \underline{\mathbb{C}}^n$. This defines an equivalence relation.

We note that only vector bundles of equal dimension can be stably isomorphic.

The following lemma tells us, that direct sum of vector bundles gives us again an addition on equivalence classes.

Lemma 2.7. *If $E_1 \cong_s E_1'$ and $E_2 \cong_s E_2'$, then $E_1 \oplus E_2 \cong_s E_1' \oplus E_2'$, i.e. $\oplus$ is well-defined on $\cong_s$-equivalence classes.*

Proof. We have $E_1 \oplus \underline{\mathbb{C}}^n \cong E_1' \oplus \underline{\mathbb{C}}^n$ and $E_2 \oplus \underline{\mathbb{C}}^m \cong E_2' \oplus \underline{\mathbb{C}}^m$. This implies $E_1 \oplus \underline{\mathbb{C}}^n \oplus E_2 \oplus \underline{\mathbb{C}}^m \cong E_1' \oplus \underline{\mathbb{C}}^n \oplus E_2' \oplus \underline{\mathbb{C}}^m$ and hence $E_1 \oplus E_2 \oplus \underline{\mathbb{C}}^{n+m} \cong E_1' \oplus E_2' \oplus \underline{\mathbb{C}}^{n+m}$ which gives the conclusion. □

We will soon need the following useful cancellation property.

Lemma 2.8. *Let E_1, E_2 and E_3 be vector bundles over X such that $E_1 \oplus E_2 \cong_s E_1 \oplus E_3$. Then we already have $E_2 \cong_s E_3$.*

Proof. Using Lemma 2.3, we obtain a vector bundle E_1' such that $E_1 \oplus E_1' \cong \underline{\mathbb{C}}^n$. Thus, adding E_1' to the equation in the statement, we obtain $\underline{\mathbb{C}}^n \oplus E_2 \cong_s \underline{\mathbb{C}}^n \oplus E_3$ and hence $E_2 \cong_s E_3$. □

We would now again like to have a group consisting of the $\cong_s$-equivalence classes under the operation $\oplus$. However, no equivalence class of a vector bundle E with dimension not constant 0 can have an inverse under $\oplus$. This is the case since adding a vector bundle to E can only increase the dimension and thus never reach the neutral element $[\underline{\mathbb{C}}^0]$.

Using the following construction, we formally adjoin inverses of all vector bundles to obtain a group. This is due to Grothendieck, and hence also known as the *Grothedieck group construction.* Since we do not use any special properties of vector bundles, only the cancellation property, it also works in greater generality.

Lemma 2.9. *Consider pairs (E_1, E_2) of vector bundles over X. Set $(E_1, E_2) = (E_1', E_2')$ if $E_1 \oplus E_2' \cong_s E_1' \oplus E_2$. This defines an equivalence relation.*

Proof. The relation is reflexive and symmetric. To see that it is transitive, let $E_1 \oplus E_2' \cong_s E_1' \oplus E_2$ and $E_1' \oplus E_2'' \cong_s E_1'' \oplus E_2'$. Adding the first two equations we get $E_1 \oplus E_2' \oplus E_1' \oplus E_2'' \cong_s E_1' \oplus E_2 \oplus E_1'' \oplus E_2'$. Canceling $E_1' \oplus E_2'$ we obtain $E_1 \oplus E_2'' \cong_s E_1'' \oplus E_2$, which is what wanted □

We can now define the second fundamental object of our study.

Definition 2.10. We denote by $\mathrm{K}(X)$ the set of pairs (E_1, E_2) of vector bundles over X with respect to the equivalence relation from the previous lemma.

We will call an element $(E_1, E_2) \in \mathrm{K}(X)$ a *virtual vector bundle* over X and $\dim E_1|_x - \dim E_2|_x$ its *virtual dimension* over $x \in X$.

Thinking about addition of differences $(a-b)+(c-d)$ leads us to the following definition of a group structure on these virtual vector bundles.

Proposition 2.11. *We define the binary operation $+$ on $\mathrm{K}(X)$ by $(E_1, E_2) + (E_3, E_4) = (E_1 \oplus E_3, E_2 \oplus E_4)$. This is well-defined and makes $\mathrm{K}(X)$ into an abelian group with neutral element $(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$ and inverses given by switching the two entries.*

We will call $\mathrm{K}(X)$ the (unreduced) K-group *of X.*

Proof. We first show that $+$ is well-defined. Thus, we assume $(E_1, E_2) = (E_1', E_2')$ and $(E_3, E_4) = (E_3', E_4')$, i.e. $E_1 \oplus E_2' \cong_s E_1' \oplus E_2$ and $E_3 \oplus E_4' \cong_s E_3' \oplus E_4$. Adding those, we obtain $E_1 \oplus E_3 \oplus E_2' \oplus E_4' \cong_s E_1' \oplus E_3' \oplus E_2 \oplus E_4$ which is equivalent to $(E_1 \oplus E_3, E_2 \oplus E_4) = (E_1' \oplus E_3', E_2' \oplus E_4')$ which we wanted to show.

Associativity, commutativity and that $(\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$ constitutes a neutral element follow from the corresponding statements for $\oplus$. The inverse element is given by switching the entries since we have $(E_1 \oplus E_2, E_2 \oplus E_1) = (\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$. □

We can think about the pair (E_1, E_2) as a formal difference $E_1 - E_2$ (a notation we will also sometimes use). This makes sense, since $(E_1, E_2) = (E_1, \underline{\mathbb{C}}^0) - (E_2, \underline{\mathbb{C}}^0)$ and $E \mapsto (E, \underline{\mathbb{C}}^0)$ is the canonical monoid homomorphism $\mathrm{Vect}(X)/\cong_s \to \mathrm{K}(X)$.

To close this subsection, we will now prove a useful lemma we will need later.

Lemma 2.12. *Every element of* $\mathrm{K}(X)$ *can be written as* $(E, \underline{\mathbb{C}}^n)$ *for some vector bundle* E *and* $n \in \mathbb{N}_0$.

Proof. Let $(E_1, E_2) \in \mathrm{K}(X)$. By Lemma 2.3, there exists a vector bundle E_2' such that $E_2 \oplus E_2' \cong \underline{\mathbb{C}}^n$. Thus,

$$(E_1, E_2) = (E_1, E_2) + (E_2', E_2') = (E_1 \oplus E_2', \underline{\mathbb{C}}^n)$$

□

2.3 The ring structure on $\mathrm{K}(X)$

We now want to transfer more of the structure of vector bundles to our groups. In particular, we want to use the tensor product to obtain a product on both $\mathrm{K}(X)$ and $\widetilde{\mathrm{K}}(X)$. Since we will need the ring structure of $\mathrm{K}(X)$ to obtain the one of $\widetilde{\mathrm{K}}(X)$, we start with the unreduced version.

Proposition 2.13. *The multiplication*

$$(E_1, E_2) \cdot (E_3, E_4) = (E_1 \otimes E_3 \oplus E_2 \otimes E_4, E_1 \otimes E_4 \oplus E_2 \otimes E_3)$$

makes $\mathrm{K}(X)$ *into a commutative ring with unit element given by* $(\underline{\mathbb{C}}^1, \underline{\mathbb{C}}^0)$.

Proof. First, we show that tensoring is compatible with stable isomorphisms. Assume $E_1 \cong_s E_2$, hence $E_1 \oplus \underline{\mathbb{C}}^n \cong E_2 \oplus \underline{\mathbb{C}}^n$. Tensoring with E' yields $E_1 \otimes E' \oplus \underline{\mathbb{C}}^n \otimes E' \cong E_2 \otimes E' \oplus \underline{\mathbb{C}}^n \otimes E'$. This equation thus also holds for $\cong_s$ where we can apply the cancellation property to obtain $E_1 \otimes E' \cong_s E_2 \otimes E'$.

Next, we show that the multiplication is actually well-defined. So assume $(E_1, E_2) = (E_1', E_2')$, i.e. $E_1 \oplus E_2' \cong E_1' \oplus E_2$. Tensoring the equation with E_3 respectively E_4 and adding the resulting equations (left hand side to right hand side and vice versa) we obtain

$$\begin{aligned} & E_1 \otimes E_3 \oplus E_2' \otimes E_3 \oplus E_1' \otimes E_4 \oplus E_2 \otimes E_4 \\ \cong_s\ & E_1' \otimes E_3 \oplus E_2 \otimes E_3 \oplus E_1 \otimes E_4 \oplus E_2' \otimes E_4 \end{aligned}$$

which, after reordering, becomes the definition of

$$\begin{aligned}&(E_1 \otimes E_3 \oplus E_2 \otimes E_4, E_1 \otimes E_4 \oplus E_2 \otimes E_3)\\ = &(E_1' \otimes E_3 \oplus E_2' \otimes E_4, E_1' \otimes E_4 \oplus E_2' \otimes E_3)\end{aligned}$$

If both $(E_1, E_2) = (E_1', E_2')$ and $(E_3, E_4) = (E_3', E_4')$, then we get $(E_1, E_2) \cdot (E_3, E_4) = (E_1', E_2') \cdot (E_3, E_4) = (E_1', E_2') \cdot (E_3', E_4')$, using the last computation and the symmetry of the definition of the multiplication. Thus, we have a well-defined map $\mathrm{K}(X) \times \mathrm{K}(X) \to \mathrm{K}(X)$.

Commutativity also follows from the symmetry of the definition. Associativity follows from the following computation

$$\begin{aligned}&(E_1 \otimes E_3 \oplus E_2 \otimes E_4, E_1 \otimes E_4 \oplus E_2 \otimes E_3) \cdot (E_5, E_6)\\ = &(E_1 \otimes E_3 \otimes E_5 \oplus E_2 \otimes E_4 \otimes E_5 \oplus E_1 \otimes E_4 \otimes E_6 \oplus E_2 \otimes E_3 \otimes E_6,\\ &\ E_1 \otimes E_3 \otimes E_6 \oplus E_2 \otimes E_4 \otimes E_6 \oplus E_1 \otimes E_4 \otimes E_5 \oplus E_2 \otimes E_3 \otimes E_5)\\ = &(E_1, E_2) \cdot (E_3 \otimes E_5 \oplus E_4 \otimes E_6, E_3 \otimes E_6 \oplus E_4 \otimes E_5)\end{aligned}$$

We have $(E_1, E_2) \cdot (\underline{\mathbb{C}}^1, \underline{\mathbb{C}}^0) = (E_1 \otimes \underline{\mathbb{C}}^1, E_2 \otimes \underline{\mathbb{C}}^1) = (E_1, E_2)$. Thus $(\underline{\mathbb{C}}^1, \underline{\mathbb{C}}^0)$ constitutes the unit element.

One last computation shows distributivity

$$\begin{aligned}&(E_1, E_2) \cdot ((E_3, E_4) + (E_5, E_6))\\ = &(E_1 \otimes (E_3 \oplus E_5) \oplus E_2 \otimes (E_4 \oplus E_6), E_1 \otimes (E_4 \oplus E_6) \oplus E_2 \otimes (E_3 \oplus E_5))\\ = &(E_1, E_2) \cdot (E_3, E_4) + (E_1, E_2) \cdot (E_5, E_6)\end{aligned}$$

□

As usual, we will mostly omit the symbol "·" and just write $(E_1, E_2)(E_3, E_4)$. We will now do the first computation of a ring $\mathrm{K}(X)$.

Lemma 2.14. *Let* pt *be the one point space. Then the map*

$$\Psi \colon \mathrm{K}(\mathrm{pt}) \longrightarrow \mathbb{Z}, \quad (E_1, E_2) \longmapsto \dim (E_1)_{\mathrm{pt}} - \dim (E_2)_{\mathrm{pt}}$$

is a well-defined ring isomorphism.

Proof. Every vector bundle over a point is trivial. Thus every pair of vector bundles over pt is of the form $(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^m)$.

Hence Ψ is given by $(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^m) \mapsto n - m$. It is well-defined since $(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^m) = (\underline{\mathbb{C}}^{n'}, \underline{\mathbb{C}}^{m'})$ implies $n + m' = n' + m$ and thus $n - m = n' - m'$. It is compatible with addition as

$$\begin{aligned}(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^m) + (\underline{\mathbb{C}}^{n'}, \underline{\mathbb{C}}^{m'}) &= (\underline{\mathbb{C}}^{n+n'}, \underline{\mathbb{C}}^{m+m'})\\ &\mapsto (n + n') - (m + m') = (n - m) + (n' - m')\end{aligned}$$

It sends the unit to the unit since $(\underline{\mathbb{C}}^1, \underline{\mathbb{C}}^0) \mapsto 1$. It is compatible with multiplication as

$$\begin{aligned}(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^m)(\underline{\mathbb{C}}^{n'}, \underline{\mathbb{C}}^{m'}) &= (\underline{\mathbb{C}}^{nn'+mm'}, \underline{\mathbb{C}}^{nm'+mn'})\\ &\mapsto (nn' + mm') - (nm' + mn') = (n - m)(n' - m')\end{aligned}$$

It is surjective and its kernel is given by elements of the form $(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^n) = (\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$ and hence is trivial. Thus the map is an isomorphism. □

2.4 Functoriality

Proposition 2.15. *Let $f\colon X \to Y$ be a map of spaces. We set*

$$\widetilde{\mathrm{K}}(f)\colon \widetilde{\mathrm{K}}(Y) \longrightarrow \widetilde{\mathrm{K}}(X), \quad [E] \longmapsto [f^*E]$$
$$\mathrm{K}(f)\colon \mathrm{K}(Y) \longrightarrow \mathrm{K}(X), \quad (E_1, E_2) \longmapsto (f^*E_1, f^*E_2)$$

With these definitions $\mathrm{K}(-)$ *and* $\widetilde{\mathrm{K}}(-)$ *become homotopy invariant, contravariant functors from compact Hausdorff spaces to abelian groups.*

Proof. We first need to see that the maps $\mathrm{K}(f)$ and $\widetilde{\mathrm{K}}(f)$ are well-defined. If $E \sim E'$, hence $E \oplus \underline{\mathbb{C}}^n \cong E' \oplus \underline{\mathbb{C}}^{n'}$, we have

$$f^*E \oplus \underline{\mathbb{C}}^n \cong f^*(E \oplus \underline{\mathbb{C}}^n) \cong f^*(E' \oplus \underline{\mathbb{C}}^{n'}) \cong f^*E' \oplus \underline{\mathbb{C}}^{n'}$$

and thus $f^*E \sim f^*E'$. Similarly, if $(E_1, E_2) = (E_1', E_2')$, hence $E_1 \oplus E_2' \oplus \underline{\mathbb{C}}^n \cong E_1' \oplus E_2 \oplus \underline{\mathbb{C}}^n$, we have

$$\begin{aligned} f^*E_1 \oplus f^*E_2' \oplus \underline{\mathbb{C}}^n &\cong f^*(E_1 \oplus E_2' \oplus \underline{\mathbb{C}}^n) \\ &\cong f^*(E_1' \oplus E_2 \oplus \underline{\mathbb{C}}^n) \cong f^*E_1' \oplus f^*E_2 \oplus \underline{\mathbb{C}}^n \end{aligned}$$

and thus $(f^*E_1, f^*E_2) = (f^*E_1', f^*E_2')$.

Since X is compact and hence paracompact, if $f \simeq g$ are homotopic, the bundles f^*E and g^*E are isomorphic (cf. [HaK, Theorem 1.6]). This implies the homotopy invariance.

Since $\mathrm{id}^*E \cong E$, we have that identities map to identities. As $(f \circ g)^*E \cong g^*(f^*E)$, we have that both of our definitions respect composition. □

We will write f^* for the maps $\mathrm{K}(f)$ and $\widetilde{\mathrm{K}}(f)$. Naturally, we also want the induced maps to be compatible with our ring structure on $\mathrm{K}(X)$.

Proposition 2.16. *Let $f\colon X \to Y$ be a map of spaces. Then the induced map* $\mathrm{K}(f)$ *is a ring homomorphism. Thus* $\mathrm{K}(-)$ *can also be considered to be a functor to rings.*

Proof. The map $\mathrm{K}(f)$ sends the unit $(\underline{\mathbb{C}}^1, \underline{\mathbb{C}}^0)$ to the unit $(\underline{\mathbb{C}}^1, \underline{\mathbb{C}}^0)$. Since pulling back vector bundles commutes with taking direct sums and tensor products, it also follows that $\mathrm{K}(f)$ is compatible with the multiplication. □

2.5 Relation between $\mathrm{K}(X)$ and $\widetilde{\mathrm{K}}(X)$

We will now establish a splitting $\mathrm{K}(X) \cong \widetilde{\mathrm{K}}(X) \oplus \mathbb{Z}$. This will lead us to the definition of the ring structure on $\widetilde{\mathrm{K}}(X)$.

Lemma 2.17. *The map*

$$\Phi\colon \mathrm{K}(X) \longrightarrow \widetilde{\mathrm{K}}(X), \quad (E, \underline{\mathbb{C}}^n) \longmapsto [E]$$

is well-defined and a natural group homomorphism. Furthermore, it is surjective and $\ker \Phi \cong \mathbb{Z}$.

Proof. By Lemma 2.12, we can write every element of $\mathrm{K}(X)$ in the form $(E, \underline{\mathbb{C}}^n)$. The map Φ is well-defined because $(E, \underline{\mathbb{C}}^n) = (E', \underline{\mathbb{C}}^{n'})$ implies $E \oplus \underline{\mathbb{C}}^{n'} \cong_s E' \oplus \underline{\mathbb{C}}^n$, thus $E \oplus \underline{\mathbb{C}}^{n'} \oplus \underline{\mathbb{C}}^m \cong E' \oplus \underline{\mathbb{C}}^n \oplus \underline{\mathbb{C}}^m$, and hence $E \sim E'$. It is compatible with the addition, by definition of the group structure of $\mathrm{K}(X)$ and $\widetilde{\mathrm{K}}(X)$. The naturality follows directly from the fact that the pullback of a trivial bundle is trivial.

The map is surjective by definition. Its kernel is given by the elements $(E, \underline{\mathbb{C}}^n)$ where $E \sim \underline{\mathbb{C}}^0$. This implies $E \oplus \underline{\mathbb{C}}^m \cong \underline{\mathbb{C}}^0 \oplus \underline{\mathbb{C}}^{m'}$ and thus

$$(E, \underline{\mathbb{C}}^n) = (E, \underline{\mathbb{C}}^0) + (\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^n) = (\underline{\mathbb{C}}^{m'}, \underline{\mathbb{C}}^m) + (\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^n) = (\underline{\mathbb{C}}^{m'}, \underline{\mathbb{C}}^{m+n})$$

Hence the kernel consists of the elements of the form $(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^m)$ for some $n, m \in \mathbb{N}_0$. This subgroup is isomorphic to $\mathbb{Z}$ under an isomorphism defined exactly as in Lemma 2.14. □

Lemma 2.18. *Let X be a space and $x_0 \in X$. Then there is a natural, naturally split short exact sequence of abelian groups*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathrm{K}}(X) & \longrightarrow & \mathrm{K}(X) & \xrightarrow{\iota^*} & \mathrm{K}(x_0) & \longrightarrow & 0 \\ & & & & & & \Psi \big\downarrow \cong & & \\ & & & & & & \mathbb{Z} & & \end{array}$$

where ι is the inclusion of x_0 into X and the map Ψ is the isomorphism from Lemma 2.14. We also obtain a natural group isomorphism

$$\widetilde{\mathrm{K}}(X) \oplus \mathbb{Z} \cong \widetilde{\mathrm{K}}(X) \oplus \mathrm{K}(x_0) \xrightarrow{\cong} \mathrm{K}(X)$$

where the map $\mathrm{K}(x_0) \to \mathrm{K}(X)$ is induced by collapsing X to its basepoint.

Proof. The composition of $\Psi \circ \iota^*$ is given by mapping (E_1, E_2) to $\dim (E_1)_{x_0} - \dim (E_2)_{x_0}$. Hence its kernel consists of pairs (E_1, E_2) such that their dimensions agree over x_0. Using Lemma 2.12, this is equal to the subgroup of elements $(E, \underline{\mathbb{C}}^n)$ such that $\dim E_{x_0} = n$. This, in turn, is isomorphic to $\widetilde{\mathrm{K}}(X)$ via the surjective map Φ from the previous lemma since the only element of $\ker \Phi$ contained in that subgroup is $(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^n) = (\underline{\mathbb{C}}^0, \underline{\mathbb{C}}^0)$ and it is hence also injective.

We obtain the desired short exact sequence. The map $c^* \colon \mathrm{K}(x_0) \to \mathrm{K}(X)$ induced by the constant map $c \colon X \to x_0$ is a split for it since $c \circ \iota = \mathrm{id}$ already holds for spaces. Hence we get the splitting $\mathrm{K}(X) \cong \widetilde{\mathrm{K}}(X) \oplus \mathrm{K}(x_0)$ as claimed.

The sequence and its split are natural since all occurring maps are. □

Remark 2.19. Note that, in the composition $\mathrm{K}(X) \xrightarrow{\iota^*} \mathrm{K}(x_0) \xrightarrow{\Psi} \mathbb{Z}$, the point x_0 is only used to determine the dimension of fibers over it, which are locally constant. Hence the identification $\mathrm{K}(X) \cong \widetilde{\mathrm{K}}(X) \oplus \mathbb{Z}$ actually only depends on the connected component of x_0.

This isomorphism corresponds to splitting off, from $\mathrm{K}(X)$, a $\mathbb{Z}$-factor corresponding to the trivial elements, i.e. those of the form $(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^m)$. This is the case because the bundles obtained by pulling back along $X \to x_0$ are exactly the trivial bundles.

The inclusion $\widetilde{\mathrm{K}}(X) \to \mathrm{K}(X)$ identifies $\widetilde{\mathrm{K}}(X)$ with the ideal of virtual vector bundles of virtual dimension 0 over the basepoint x_0.

Remark 2.20. Note that, using the above, we have in particular a natural isomorphism

$$\widetilde{\mathrm{K}}(X_+) \cong \ker(\mathrm{K}(X_+) \to \mathrm{K}(+)) \cong \mathrm{K}(X)$$

With this identification the short exact sequence from above becomes, for a space X and $x_0 \in X$,

$$0 \longrightarrow \widetilde{\mathrm{K}}(X) \longrightarrow \widetilde{\mathrm{K}}(X_+) \xrightarrow{(\iota_+)^*} \widetilde{\mathrm{K}}((x_0)_+) \longrightarrow 0$$

The splitting

$$\widetilde{\mathrm{K}}(X_+) \cong \widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(\mathrm{S}^0)$$

is then given by the map $\widetilde{\mathrm{K}}(\mathrm{S}^0) \to \widetilde{\mathrm{K}}(X_+)$ induced by collapsing X to a point and the map

$$\widetilde{\mathrm{K}}(X_+) \cong \ker(\mathrm{K}(X_+) \to \mathrm{K}(+)) \longrightarrow \ker(\mathrm{K}(X) \to \mathrm{K}(x_0)) \cong \widetilde{\mathrm{K}}(X)$$

given by sending a virtual vector bundle a to $a|_X - \underline{\mathbb{C}}^n$ where n is the virtual dimension of a over x_0.

2.6 The ring structure on $\widetilde{\mathrm{K}}(X)$

We would now also like to have a ring structure on $\widetilde{\mathrm{K}}(X)$. Trying the same idea as in the unreduced case (and assuming it would be well-defined) yields $\underline{\mathbb{C}}^0 \cong E \otimes \underline{\mathbb{C}}^0 \sim E \otimes \underline{\mathbb{C}}^1 \cong E$, i.e., as a ring, we would necessarily have $\widetilde{\mathrm{K}}(X) \cong 0$. This would not be very useful.

Instead we use, for a space X and $x_0 \in X$, the short exact sequence from Lemma 2.18

$$0 \longrightarrow \widetilde{\mathrm{K}}(X) \longrightarrow \mathrm{K}(X) \longrightarrow \mathrm{K}(x_0) \longrightarrow 0$$

The map $\mathrm{K}(X) \to \mathrm{K}(x_0)$ is a ring homomorphisms since it is induced from a map of spaces. Its kernel is an ideal in $\mathrm{K}(X)$ and thus inherits a (non-unital) multiplication. The multiplication on $\widetilde{\mathrm{K}}(X)$ is now defined to be the corresponding multiplication under the isomorphism $\Phi\colon \ker(\mathrm{K}(X) \to \mathrm{K}(x_0)) \to \widetilde{\mathrm{K}}(X)$. This makes $\widetilde{\mathrm{K}}(X)$ into a non-unital ring.

Since this definition depends on the point x_0, we have to work with pointed spaces when considering $\widetilde{\mathrm{K}}(X)$ as a ring.

In the following lemma, we will obtain from this definition an explicit, but cumbersome, description of this multiplication. This is only to get a feel for it, we will not need it later.

Lemma 2.21. *Let X be a pointed space, E_1 and E_2 two vector bundles over X, and n_1 respectively n_2 their dimension over x_0. Also let, for $i \in \{1,2\}$, E_i' be a vector bundle such that $E_i \oplus E_i'$ is trivial. Then the multiplication on $\widetilde{\mathrm{K}}(X)$ is given by*

$$[E_1] \cdot [E_2] = [E_1 \otimes E_2 \oplus E_1' \otimes \underline{\mathbb{C}}^{n_2} \oplus E_2' \otimes \underline{\mathbb{C}}^{n_1}]$$

Proof. The element E_i corresponds to $(E_i, \underline{\mathbb{C}}^{n_i})$ in $\ker(\mathrm{K}(X) \to \mathrm{K}(x_0))$. We have

$$\begin{aligned}(E_1, \underline{\mathbb{C}}^{n_1})(E_2, \underline{\mathbb{C}}^{n_2}) &= (E_1 \otimes E_2 \oplus \underline{\mathbb{C}}^{n_1 n_2}, E_1 \otimes \underline{\mathbb{C}}^{n_2} \oplus E_2 \otimes \underline{\mathbb{C}}^{n_1}) \\ &= (E_1 \otimes E_2 \oplus \underline{\mathbb{C}}^{n_1 n_2} \oplus E_1' \otimes \underline{\mathbb{C}}^{n_2} \oplus E_2' \otimes \underline{\mathbb{C}}^{n_1}, \underline{\mathbb{C}}^{m_1 n_2 + m_2 n_1})\end{aligned}$$

where $E_i \oplus E_i' \cong \underline{\mathbb{C}}^{m_i}$. The result follows since $\underline{\mathbb{C}}^{n_1 n_2}$ disappears in $\widetilde{\mathrm{K}}(X)$. $\square$

Naturally, we again want that this multiplication is compatible with induced maps. This is what we will show now.

Proposition 2.22. *With the just defined multiplicative structure, $\widetilde{\mathrm{K}}(-)$ becomes an homotopy invariant contravariant functor from pointed compact Hausdorff spaces to non-unital rings.*

Proof. We only need to show that induced maps respect the multiplication, everything else was shown in Proposition 2.15. One possibility to do this is through the formula in the previous proposition, using the compatibility of the pullback with everything occurring. Another, more conceptual, way is directly over the definition of the multiplication. From the naturality of the short exact sequence in Lemma 2.18 we get the following commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{K}}(Y) & \hookrightarrow & \mathrm{K}(Y) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{K}}(X) & \hookrightarrow & \mathrm{K}(X) \end{array}$$

Note that, by definition, the injections respect the multiplication. Since the map on the left is just a restriction of the map on the right, which is a ring homomorphism, it is a homomorphism of non-unital rings. $\square$

2.7 Unreduced external product and Product Theorem

We now have internal multiplications for $\mathrm{K}(X)$ as well as $\widetilde{\mathrm{K}}(X)$. But, to be able to state some of the coming theorems, we will also need an external product for our objects. They will be related in a way similar to the cross and cup product of ordinary singular cohomology.

Definition 2.23. Let $\mathrm{pr}_X \colon X \times Y \to X$ and $\mathrm{pr}_Y \colon X \times Y \to Y$ be the projections. Then we define an external product

$$* \colon \mathrm{K}(X) \otimes \mathrm{K}(Y) \longrightarrow \mathrm{K}(X \times Y)$$

via $a \otimes b \mapsto \mathrm{pr}_X^*(a) \cdot \mathrm{pr}_Y^*(b)$.

Lemma 2.24. *The external product $*$ is well-defined, natural and a ring homomorphism.*

Proof. Since pr_X^* and pr_Y^* are ring homomorphisms, we have, by the universal property of the tensor product of rings, that $*$ is a well-defined ring homomorphism.

For the naturality we want to check that, for maps $f\colon X \to X'$ and $g\colon Y \to Y'$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{K}(X') \otimes \mathrm{K}(Y') & \xrightarrow{\;*\;} & \mathrm{K}(X' \times Y') \\ {\scriptstyle f^* \otimes g^*}\downarrow & & \downarrow{\scriptstyle (f\times g)^*} \\ \mathrm{K}(X) \otimes \mathrm{K}(Y) & \xrightarrow{\;*\;} & \mathrm{K}(X \times Y) \end{array}$$

This is equivalent to

$$(f \times g)^*(\mathrm{pr}_{X'}^*(a) \cdot \mathrm{pr}_{Y'}^*(b)) = \mathrm{pr}_X^*(f^*(a)) \cdot \mathrm{pr}_Y^*(g^*(b))$$

for all $a \in \mathrm{K}(X')$ and $b \in \mathrm{K}(Y')$. Since $(f \times g)^*$ is a ring homomorphism, the left side of the equation is equal to $(f \times g)^*(\mathrm{pr}_{X'}^*(a)) \cdot (f \times g)^*(\mathrm{pr}_{Y'}^*(b))$. Because there are commuting diagrams

$$\begin{array}{ccc} X \times Y & \xrightarrow{\mathrm{pr}_X} & X \\ {\scriptstyle f\times g}\downarrow & & \downarrow{\scriptstyle f} \\ X' \times Y' & \xrightarrow{\mathrm{pr}_{X'}} & X' \end{array} \qquad\qquad \begin{array}{ccc} X \times Y & \xrightarrow{\mathrm{pr}_Y} & Y \\ {\scriptstyle f\times g}\downarrow & & \downarrow{\scriptstyle g} \\ X' \times Y' & \xrightarrow{\mathrm{pr}_{Y'}} & Y' \end{array}$$

and thus $\mathrm{pr}_{X'} \circ (f \times g) = f \circ \mathrm{pr}_X$ and $\mathrm{pr}_{Y'} \circ (f \times g) = g \circ \mathrm{pr}_Y$, we have that

$$(f \times g)^*(\mathrm{pr}_{X'}^*(a)) \cdot (f \times g)^*(\mathrm{pr}_{Y'}^*(b)) = \mathrm{pr}_X^*(f^*(a)) \cdot \mathrm{pr}_Y^*(g^*(b))$$

This finishes the proof. □

The following lemma will give us, in our case, the usual relation between internal and external multiplication.

Lemma 2.25. *Let Δ be the diagonal map $X \to X \times X$. Then the composition*

$$\mathrm{K}(X) \otimes \mathrm{K}(X) \xrightarrow{\;*\;} \mathrm{K}(X \times X) \xrightarrow{\Delta^*} \mathrm{K}(X)$$

is equal to the normal multiplication on $\mathrm{K}(X)$.

Proof. We have

$$\Delta^*(a * b) = \Delta^*(\mathrm{pr}_1^*(a) \cdot \mathrm{pr}_2^*(b)) = \Delta^*(\mathrm{pr}_1^*(a)) \cdot \Delta^*(\mathrm{pr}_2^*(b)) = ab$$

since $\mathrm{pr}_i \circ \Delta = \mathrm{id}$ for $i \in \{1, 2\}$. □

We are now almost ready to formulate the Product Theorem, which will be the main ingredient in the proof of the fundamental Bott Periodicity Theorem.

Definition 2.26. We will denote by H the canonical line bundle over $\mathbb{C}\mathrm{P}^1$.

Lemma 2.27. *We have* $(H \otimes H) \oplus \underline{\mathbb{C}}^1 \cong H \oplus H$. *In* $\mathrm{K}(\mathbb{C}\mathrm{P}^1) \cong \mathrm{K}(\mathrm{S}^2)$ *this becomes the equation* $(H-1)^2 = 0$.

Proof. Let D^2_+ respectively D^2_- be the upper respectively lower hemisphere (to be precise, we take them to be open and a little extended over the equator). Now any n-dimensional vector bundle $E \to \mathrm{S}^2$ is trivial over both D^2_+ and D^2_- since they are contractible. Hence E is completely determined by a clutching function $\mathrm{S}^1 \to \mathrm{GL}_n(\mathbb{C})$ (since $\mathrm{D}^2_+ \cap \mathrm{D}^2_- \simeq \mathrm{S}^1$). It is a classical exercise that in the case of H one such function is given by the inclusion $z \mapsto (z)$ (for $z \in \mathrm{S}^1 \subset \mathbb{C}$), see e.g. [HaK, Example 1.13].

Hence clutching functions for $(H \otimes H) \oplus \underline{\mathbb{C}}^1$ respectively $H \oplus H$ are given by

$$z \longmapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{respectively} \qquad z \longmapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

If these maps are homotopic, we are done since then the two bundles are isomorphic. The next lemma proves this in a little greater generality (so we can refer to it later). □

Lemma 2.28. *Let X be a space and $f, g\colon X \to \mathrm{GL}_n(\mathbb{C})$ two maps. Then $f \oplus g\colon X \to \mathrm{GL}_{2n}(\mathbb{C})$ and $(f \cdot g) \oplus \mathrm{const}_{\mathbb{1}_n}\colon X \to \mathrm{GL}_{2n}(\mathbb{C})$ are homotopic. Here $\cdot$ denotes pointwise matrix multiplication and $\oplus$ pointwise direct sum of matrices, which is given by*

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Proof. Since $\mathrm{GL}_{2n}(\mathbb{C})$ is path-connected, there is a path $\alpha\colon \mathrm{I} \to \mathrm{GL}_{2n}(\mathbb{C})$ between the identity matrix $\mathbb{1}_{2n}$ and the matrix

$$\begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$$

Then the map $X \times \mathrm{I} \to \mathrm{GL}_{2n}(\mathbb{C})$ given by

$$(x,t) \longmapsto \begin{pmatrix} f(x) & 0 \\ 0 & \mathbb{1}_n \end{pmatrix} \alpha(t) \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & g(x) \end{pmatrix} \alpha(t)$$

is a homotopy between

$$x \longmapsto \begin{pmatrix} f(x) & 0 \\ 0 & g(x) \end{pmatrix} \qquad \text{and} \qquad x \longmapsto \begin{pmatrix} f(x) \cdot g(x) & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}$$

as we wanted. □

Theorem 2.29 (Product Theorem)**.** *The map*

$$\mathrm{K}(X) \otimes \mathbb{Z}[H]/(H^2 - 1) \longrightarrow \mathrm{K}(X \times \mathrm{S}^2)$$

given by $a \otimes b \mapsto a * b$ is an isomorphism of rings for all X.

Sketch of proof. The basic idea is to, in a few steps, understand vector bundles over $X \times \mathrm{S}^2$ in terms of vector bundles over X.

The idea of the proof of two previous lemmas was to consider a vector bundle over S^2 as a clutching function $\mathrm{S}^1 \to \mathrm{GL}_n(\mathbb{C})$. Applying the same idea in a more general case, we can glue together vector bundles over $X \times \mathrm{S}^2$ from a bundle $p\colon E \to X$ and an automorphism f of the vector bundle $p \times \mathrm{id}\colon E \times \mathrm{S}^1 \to X \times \mathrm{S}^1$ (i.e. a continuous choice of isomorphism $E_x \to E_x$ for each $x \in X$ and $z \in \mathrm{S}^1$). Namely we take the two vector bundles E_+ given by $p \times \mathrm{id}\colon E \times \mathrm{D}^2_+ \to X \times \mathrm{D}^2_+$ and E_- given by $p \times \mathrm{id}\colon E \times \mathrm{D}^2_- \to X \times \mathrm{D}^2_-$ and glue them together over $X \times \mathrm{S}^1$ via f. We can obtain every vector bundle over $X \times \mathrm{S}^2$ in this way.

Now, to prove the statement, one reduces the necessary complexity of the function f, using Analysis and Linear Algebra, until one arrives at a form which is easy to handle with respect to the map of which one wants to prove that it is an isomorphism.

The details of this proof can be found in most books on topological K-theory, e.g. [HaK, Theorem 2.2] or the classical [At, Corollary 2.2.3]. □

Corollary 2.30. *The map*

$$\mathbb{Z}[H]/(H^2 - 1) \longrightarrow \mathrm{K}(\mathrm{S}^2)$$

given by $H \mapsto H$ is an isomorphism of rings.

Proof. It is the composition

$$\mathbb{Z}[H]/(H^2 - 1) \xrightarrow{\cong} \mathbb{Z} \otimes \big(\mathbb{Z}[H]/(H^2 - 1)\big)$$
$$\cong \mathrm{K}(\mathrm{pt}) \otimes \big(\mathbb{Z}[H]/(H^2 - 1)\big) \xrightarrow{*} \mathrm{K}(\mathrm{pt} \times \mathrm{S}^2)$$

where the first map is given by $a \mapsto 1 \otimes a$ and the last map is an isomorphism by the previous theorem. □

2.8 The long exact sequence of a pair

We now start to prove cohomological properties of $\widetilde{\mathrm{K}}$.

Proposition 2.31. *Let $A \subseteq X$ be an inclusion of pointed spaces such that A is closed in X. Then the sequence of non-unital rings*

$$\widetilde{\mathrm{K}}(X/A) \xrightarrow{q^*} \widetilde{\mathrm{K}}(X) \xrightarrow{\iota^*} \widetilde{\mathrm{K}}(A)$$

induced by the inclusion $\iota\colon A \to X$ and the quotient map $q\colon X \to X/A$ is exact.

Proof. Since $A \subseteq X$ is closed, it is compact and Hausdorff; also, by Lemma 1.15, X/A is as well, so $\widetilde{\mathrm{K}}(A)$ and $\widetilde{\mathrm{K}}(X/A)$ actually make sense.

We need to show $\ker \iota^* = \operatorname{Im} q^*$. The inclusion $\ker \iota^* \supset \operatorname{Im} q^*$, i.e. $\iota^* \circ q^* = 0$, is clear since $q \circ \iota$ is constant and thus factors through a point, for which we have $\widetilde{\mathrm{K}}(\mathrm{pt}) \cong 0$.

For the other inclusion we need to show that, for every vector bundle E over X with $E|_A \sim \underline{\mathbb{C}}^0$, we have $E \sim q^*E'$ for some vector bundle E' over X/A. By adding a trivial bundle to E, we find a bundle, reduced stably isomorphic to E, which restricts to a trivial bundle over A. Thus we can, without loss of generality, assume that $E|_A$ is trivial.

We will now show that there is an E' such that $E \cong q^*E'$. To construct E' we want to collapse $E|_A$ to a single fiber. To do this, we choose a trivialization $h\colon A \times \mathbb{C}^n \to E|_A$ (without it we would not know how to identify the fibers). Then we can set $E' = E/h = E/(h(a,v) \sim h(a',v))$ and denote by $r\colon E \to E/h$ the quotient map. We want E/h to be a vector bundle. For this we only need to show that it has a local trivialization around the point A/A.

By Lemma 1.23, we can extend h to a trivialization h' on an open neighborhood U of A. This implies that E/h is trivial over the open set $q(U) \ni A/A$, using the map $g\colon (U/A) \times \mathbb{C}^n \to (E/h)|_{U/A}$ given by the factorization

$$\begin{array}{ccccc} U \times \mathbb{C}^n & \xrightarrow{\;h'\;} & E|_U & \xrightarrow{\;r|_U\;} & (E/h)|_{U/A} \\ & \searrow & & \nearrow_{g} & \\ & & (U/A) \times \mathbb{C}^n & & \end{array}$$

Since it is an isomorphism on each fiber, this is an isomorphism of vector bundles and hence a trivialization.

It only remains to show that indeed $q^*(E/h) \cong E$. For this, consider the diagram

$$\begin{CD} E @>r>> E/h \\ @VVV @VVV \\ X @>q>> X/A \end{CD}$$

The map r sits above q and is an isomorphism on each fiber. Hence, since this is exactly the defining property of the pullback, we have that the bundle E, together with the map r, constitutes the pullback of E/h under q. $\square$

In general we know that, if the inclusion $A \subseteq X$ is a cofibration and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence. Hence it induces an isomorphism on $\widetilde{\mathrm{K}}$ (when this makes sense, i.e. X compact and $A \subseteq X$ closed). It will be useful to know this in greater generality.

Lemma 2.32. *Let $A \subseteq X$ be an inclusion of pointed spaces such that A is closed in X and contractible. Then the quotient map $q\colon X \to X/A$ induces an isomorphism of non-unital rings $q^*\colon \widetilde{\mathrm{K}}(X/A) \to \widetilde{\mathrm{K}}(X)$.*

Proof. Since A is contractible, we have that $E|_A$ is trivial. Let $h\colon E|_A \to A \times \mathbb{C}^n$ be a trivialization. As in the previous proposition, we obtain a vector bundle $E/h = E/(h^{-1}(a,v) \sim h^{-1}(a',v))$. We want to see that the isomorphism class of E/h does not depend on the choice of h.

Let h_0 and h_1 be two such choices. Consider $f = h_1 \circ h_0^{-1} \colon A \times \mathbb{C}^n \to A \times \mathbb{C}^n$. This gives us a function $\phi \colon A \to \mathrm{GL}_n(\mathbb{C})$, via $\phi(a) = (f|_{\{a\}\times\mathbb{C}^n} \colon \mathbb{C}^n \to \mathbb{C}^n)$, such that $h_1(e) = (p(e), \phi(p(e)) \cdot (\mathrm{pr}_{\mathbb{C}^n} \circ h_0)(e))$, where $p \colon E|_A \to A$ is the bundle map. Since A is contractible and $\mathrm{GL}_n(\mathbb{C})$ is path-connected, there is a homotopy $\Phi \colon A \times \mathrm{I} \to \mathrm{GL}_n(\mathbb{C})$ between $\mathrm{const}_{\mathbb{1}}$ and ϕ. From this we obtain a homotopy $H \colon E|_A \times \mathrm{I} \to A \times \mathbb{C}^n$ between h_0 and h_1 given by $H(e,t) = (p(e), \Phi(p(e),t) \cdot (\mathrm{pr}_{\mathbb{C}^n} \circ h_0)(e))$. This map gives us, in turn, a map $g \colon E|_A \times \mathrm{I} \to A \times \mathbb{C}^n \times \mathrm{I} \cong A \times \mathrm{I} \times \mathbb{C}^n$ given by $g(e,t) = (H(e,t),t)$. This is a trivialization over $A \times \mathrm{I}$ of the bundle $E \times \mathrm{I}$ over $X \times \mathrm{I}$ (since it is a fiberwise isomorphism and hence an isomorphism of vector bundles).

We now set $F = (E \times \mathrm{I})/(g^{-1}(a,t,v) \sim g^{-1}(a',t,v))$. This corresponds to contracting $E|_A \times \{t\} \subset E \times \mathrm{I}$ to a single fiber for each $t \in \mathrm{I}$. Hence it comes with a map $F \to (X/A) \times \mathrm{I}$. Note that, when restricted to $(X/A) \times \{0\}$, this is just the bundle E/h_0 and analogously for restricting to $(X/A) \times \{1\}$ and E/h_1. Hence, since $X/A \times \mathrm{I}$ is compact and hence paracompact, F being a vector bundle would already imply $E/h_0 \cong E/h_1$.

To see that it is one, we only need to show that it has trivializations around all points in $(A/A) \times \mathrm{I}$. Similarly to the last proof, we can, by Lemma 1.23, extend the trivialization g of $E|_A \times \mathrm{I}$ to a trivialization $g' \colon E|_U \times \mathrm{I} \to U \times \mathrm{I} \times \mathbb{C}^n$ over an open set $U \subseteq X$ containing $A \times \mathrm{I}$. The following factorization then gives us a trivialization of F around $(A/A) \times \mathrm{I}$

$$\begin{array}{ccccc} E|_U \times \mathrm{I} & \xrightarrow{g'} & U \times \mathrm{I} \times \mathbb{C}^n & \longrightarrow & (U/A) \times \mathrm{I} \times \mathbb{C}^n \\ & \searrow & & \nearrow & \\ & & F|_{U/A} & & \end{array}$$

Hence we have proven that the bundle E/h over X/A does not depend on the choice of the trivialization h. Thus we get a well-defined map (of sets) $\psi \colon \mathrm{Vect}(X) \to \mathrm{Vect}(X/A)$.

We claim that this is an inverse to $q^* \colon \mathrm{Vect}(X/A) \to \mathrm{Vect}(X)$. The composition $q^* \circ \psi$ being the identity follows from $q^*(E/h) \cong E$, which we have shown in the last part of the proof of the previous proposition. To see that $\psi \circ q^* = \mathrm{id}$, consider a vector bundle E over X/A. A choice of isomorphism $E_{A/A} \cong \mathbb{C}^n$ gives us a trivialization h over A of the pullback bundle q^*E. Then the map $q^*E \to E$ factors through the quotient map $q^*E \to (q^*E)/h$ and the resulting map $(q^*E)/h \to E$ is an isomorphism.

Hence $q^* \colon \mathrm{Vect}(X/A) \to \mathrm{Vect}(X)$ is bijective and thus $q^* \colon \widetilde{\mathrm{K}}(X/A) \to \widetilde{\mathrm{K}}(X)$ surjective. It is also injective since $q^*([E]) = [\underline{\mathbb{C}}^0]$ implies

$$q^*(E \oplus \underline{\mathbb{C}}^n) \cong q^*E \oplus \underline{\mathbb{C}}^n \cong \underline{\mathbb{C}}^m \cong q^*(\underline{\mathbb{C}}^m)$$

and hence $E \oplus \underline{\mathbb{C}}^n \cong \underline{\mathbb{C}}^m$, i.e. $E \sim \underline{\mathbb{C}}^0$. Since $q^* \colon \widetilde{\mathrm{K}}(X/A) \to \widetilde{\mathrm{K}}(X)$ is a bijective homomorphism of non-unital rings, it is an isomorphism. □

We now want to extend the exact sequence $\widetilde{\mathrm{K}}(X/A) \to \widetilde{\mathrm{K}}(X) \to \widetilde{\mathrm{K}}(A)$ to the left. This will be done by a method that applies more generally in similar situations, namely by using the so called Puppe sequence.

Proposition 2.33. *Let (X, A) be a pair of pointed spaces with $A \subseteq X$ closed. Then there is a natural long exact sequence of non-unital rings*

$$\begin{aligned} \cdots \longrightarrow \widetilde{\mathrm{K}}(\Sigma^k(X/A)) \longrightarrow \widetilde{\mathrm{K}}(\Sigma^k X) \longrightarrow \widetilde{\mathrm{K}}(\Sigma^k A) \\ \longrightarrow \widetilde{\mathrm{K}}(\Sigma^{k-1}(X/A)) \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{K}}(A) \end{aligned}$$

Proof. Consider the following, commutative up to homotopy, diagram

$$\begin{array}{ccccc}
A & & & & \\
\downarrow & & & & \\
X & & & & \\
\downarrow & & & & \\
X \cup \widetilde{\mathrm{C}}A & \longrightarrow & X/A & & \\
\downarrow & & & & \\
(X \cup \widetilde{\mathrm{C}}A) \cup \widetilde{\mathrm{C}}X & \longrightarrow & \Sigma A & & \\
\downarrow & & \downarrow & & \\
((X \cup \widetilde{\mathrm{C}}A) \cup \widetilde{\mathrm{C}}X) \cup \widetilde{\mathrm{C}}(X \cup \widetilde{\mathrm{C}}A) & \longrightarrow & \Sigma X & & \\
 & & \downarrow & & \\
 & & \Sigma X \cup \widetilde{\mathrm{C}}(\Sigma A) & \longrightarrow & \Sigma X/\Sigma A \\
 & & \downarrow & & \\
 & & \vdots & &
\end{array}$$

where the vertical maps are given by inclusion into the reduced mapping cone of the previous map and the horizontal maps by quotienting out the reduced cone added in that line. By Lemma 1.15 and Lemma 1.16, all spaces occurring are compact and Hausdorff. In particular it makes sense to say that, by Lemma 2.32, the horizontal maps induce isomorphisms on $\widetilde{\mathrm{K}}$.

We now want to see that the square in the diagram commutes up to homotopy. One of the two compositions $(X \cup \widetilde{\mathrm{C}}A) \cup \widetilde{\mathrm{C}}X \to \Sigma X$ is given by collapsing $X \cup \widetilde{\mathrm{C}}A$ to a point. Let us call this map f. The other composition, which we will call g, is given by collapsing $\widetilde{\mathrm{C}}X$ to a point and then including the resulting ΣA into ΣX. There is a homotopy from g to f given by slowly expanding the collapsed $\widetilde{\mathrm{C}}X$ until it is mapped surjectively to ΣX and $X \cup \widetilde{\mathrm{C}}A$ is collapsed.

Thus the diagram gives us the following sequence

$$\begin{aligned} \cdots \longrightarrow \widetilde{\mathrm{K}}(\Sigma^2 A) \longrightarrow \widetilde{\mathrm{K}}(\Sigma X/\Sigma A) \longrightarrow \widetilde{\mathrm{K}}(\Sigma X) \longrightarrow \widetilde{\mathrm{K}}(\Sigma A) \\ \longrightarrow \widetilde{\mathrm{K}}(X/A) \longrightarrow \widetilde{\mathrm{K}}(X) \longrightarrow \widetilde{\mathrm{K}}(A) \end{aligned}$$

which is exact, since taking a map down and the composition of the next map down with the corresponding map to the right is exactly a sequence as in Proposition 2.31 (here we use that in a reduced mapping cone $X \cup \widetilde{\mathrm{C}}A$ the subspace X is closed). Because of Lemma 1.18, we can replace the terms of the form $\widetilde{\mathrm{K}}(\Sigma^k X/\Sigma^k A)$ with $\widetilde{\mathrm{K}}(\Sigma^k(X/A))$.

The resulting long exact sequence is natural, since all constructions we applied to obtain it are. □

We can now draw two corollaries from the existence of this long exact sequence.

Lemma 2.34. *Let X and Y be pointed spaces. Then there is a natural group isomorphism*

$$\widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(Y) \xrightarrow{\cong} \widetilde{\mathrm{K}}(X \vee Y)$$

given by the collapsing maps $X \vee Y \to X$ and $X \vee Y \to Y$.

Proof. Consider the long exact sequence for the pair $(X \vee Y, Y)$.

$$\cdots \longrightarrow \widetilde{\mathrm{K}}(\Sigma Y) \longrightarrow \widetilde{\mathrm{K}}((X \vee Y)/Y) \longrightarrow \widetilde{\mathrm{K}}(X \vee Y) \longrightarrow \widetilde{\mathrm{K}}(Y)$$

The last map naturally splits via the map induced by $X \vee Y \to Y$ given by collapsing X to the basepoint and thus is surjective. The map $\widetilde{\mathrm{K}}(\Sigma Y) \to \widetilde{\mathrm{K}}((X \vee Y)/Y)$ is given by the diagram

$$\begin{array}{ccc} (X \vee Y) \cup \widetilde{\mathrm{C}}Y & \longrightarrow & (X \vee Y)/Y \\ \downarrow f & & \\ ((X \vee Y) \cup \widetilde{\mathrm{C}}Y) \cup \widetilde{\mathrm{C}}(X \vee Y) & \xrightarrow{q} & \Sigma Y \end{array}$$

and thus is zero, since a vector bundle pulled back along $q \circ f$ is trivial on $X \subset X \vee \widetilde{\mathrm{C}}Y \cong (X \vee Y) \cup \widetilde{\mathrm{C}}Y$ (as this subset is collapsed by the map) and also on $\widetilde{\mathrm{C}}Y \subset X \vee \widetilde{\mathrm{C}}Y$ because it is contractible. Here we used that $(X \vee Y) \cup \widetilde{\mathrm{C}}Y \cong X \vee \widetilde{\mathrm{C}}Y$.

Hence the last three terms of the long exact sequence split off to form the natural, naturally split short exact sequence

$$0 \longrightarrow \widetilde{\mathrm{K}}(X) \longrightarrow \widetilde{\mathrm{K}}(X \vee Y) \longrightarrow \widetilde{\mathrm{K}}(Y) \longrightarrow 0$$

which gives us a natural isomorphism $\widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(Y) \cong \widetilde{\mathrm{K}}(X \vee Y)$ as claimed. □

Lemma 2.35. *Let X and Y be pointed spaces. There is a natural, naturally split short exact sequence of groups*

$$0 \longrightarrow \widetilde{\mathrm{K}}(X \wedge Y) \longrightarrow \widetilde{\mathrm{K}}(X \times Y) \longrightarrow \widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(Y) \longrightarrow 0$$

where the first map is induced by the quotient map and the second map by the inclusions of $X \times \{y_0\}$ and $\{x_0\} \times Y$ into $X \times Y$.

We also obtain a natural group isomorphism

$$\widetilde{\mathrm{K}}(X \wedge Y) \oplus \widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(Y) \xrightarrow{\cong} \widetilde{\mathrm{K}}(X \times Y)$$

given by the quotient map and the two projections. Furthermore, the composition of its inverse with the projections onto $\widetilde{\mathrm{K}}(X)$ *respectively* $\widetilde{\mathrm{K}}(Y)$ *are given by the inclusions of* $X \times \{y_0\}$ *respectively* $\{x_0\} \times Y$ *into* $X \times Y$*.*

Proof. We consider the end of the long exact sequence for the pair $(X \times Y, X \vee Y)$

$$\begin{array}{ccccccccc} \widetilde{\mathrm{K}}(\Sigma(X \times Y)) & \to & \widetilde{\mathrm{K}}(\Sigma(X \vee Y)) & \to & \widetilde{\mathrm{K}}(X \wedge Y) & \to & \widetilde{\mathrm{K}}(X \times Y) & \to & \widetilde{\mathrm{K}}(X \vee Y) \\ & & \wr\| & & & & & & \wr\| \\ & & \widetilde{\mathrm{K}}(\Sigma X \vee \Sigma Y) & & & & & & \widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(Y) \\ & & \wr\| & & & & & & \\ & & \widetilde{\mathrm{K}}(\Sigma X) \oplus \widetilde{\mathrm{K}}(\Sigma Y) & & & & & & \end{array}$$

where we use Lemma 1.17 and Lemma 2.34 for the isomorphisms.

The rightmost map splits (and thus is surjective) via

$$\begin{aligned} \widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(Y) &\longrightarrow \widetilde{\mathrm{K}}(X \times Y) \\ (a, b) &\longmapsto \mathrm{pr}_X^*(a) + \mathrm{pr}_Y^*(b) \end{aligned}$$

where pr_X and pr_Y are the projections. This is a split since $\mathrm{pr}_X^*(a)$ is trivial on $\{x_0\} \times Y$, as $\mathrm{pr}_X(\{x_0\} \times Y)$ is a point, and analogously $\mathrm{pr}_Y^*(b)$ is trivial on $X \times \{y_0\}$.

The leftmost map is also surjectively split via

$$\begin{aligned} \widetilde{\mathrm{K}}(\Sigma X) \oplus \widetilde{\mathrm{K}}(\Sigma Y) &\longrightarrow \widetilde{\mathrm{K}}(\Sigma(X \times Y)) \\ (a, b) &\longmapsto (\Sigma \mathrm{pr}_X)^*(a) + (\Sigma \mathrm{pr}_Y)^*(b) \end{aligned}$$

The reason is again that $(\Sigma \mathrm{pr}_X)(\Sigma(\{x_0\} \times Y))$ is $\Sigma\{x_0\} \subseteq \Sigma X$, i.e. a point (and analogously for Y). Therefore the second map in the sequence is zero.

Hence the last three terms split off to form a split short exact sequence. Since everything we did here is natural, the sequence and the split are also natural. □

2.9 Reduced external product and Bott Periodicity

We now want to define an external product $\widetilde{\mathrm{K}}(X) \otimes \widetilde{\mathrm{K}}(Y) \to \widetilde{\mathrm{K}}(X \wedge Y)$. We will derive it from the unreduced version using the results from Section 2.5.

Proposition 2.36. *Let* X *and* Y *be pointed spaces. Then the external product* $*\colon \mathrm{K}(X) \otimes \mathrm{K}(Y) \to \mathrm{K}(X \times Y)$ *restricts (in a sense made precise in the proof below) to a natural homomorphism of non-unital rings*

$$\star\colon \widetilde{\mathrm{K}}(X) \otimes \widetilde{\mathrm{K}}(Y) \to \widetilde{\mathrm{K}}(X \wedge Y)$$

Furthermore, for fixed pointed spaces X *and* Y*,* $\star$ *is an isomorphism if and only if* $*$ *is.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{K}(X) \otimes \mathrm{K}(Y) & \xrightarrow{\quad\quad *\quad\quad} & \mathrm{K}(X \times Y) \\
\uparrow \cong & & \uparrow \cong \\
(\widetilde{\mathrm{K}}(X) \oplus \mathrm{K}(x_0)) \otimes (\widetilde{\mathrm{K}}(Y) \oplus \mathrm{K}(y_0)) & \longrightarrow & \widetilde{\mathrm{K}}(X \times Y) \oplus \mathrm{K}((x_0, y_0)) \\
\uparrow \cong & & \uparrow \cong \\
\widetilde{\mathrm{K}}(X) \otimes \widetilde{\mathrm{K}}(Y) & \longrightarrow & \widetilde{\mathrm{K}}(X \wedge Y) \\
\oplus & & \oplus \\
\widetilde{\mathrm{K}}(X) \otimes \mathrm{K}(y_0) & \xrightarrow{\cong} & \widetilde{\mathrm{K}}(X) \\
\oplus & & \oplus \\
\mathrm{K}(x_0) \otimes \widetilde{\mathrm{K}}(Y) & \xrightarrow{\cong} & \widetilde{\mathrm{K}}(Y) \\
\oplus & & \oplus \\
\mathrm{K}(x_0) \otimes \mathrm{K}(y_0) & \xrightarrow{\cong} & \mathrm{K}((x_0, y_0))
\end{array}
$$

where the upper two vertical isomorphisms come from Lemma 2.18 and the lower right vertical map is the isomorphism from Lemma 2.35. We will now show that the four horizontal maps on the bottom actually exist in this form.

Since the images of $\mathrm{K}(x_0) \to \mathrm{K}(X)$ and $\mathrm{K}(y_0) \to \mathrm{K}(Y)$ are exactly the elements $(\underline{\mathbb{C}}^n, \underline{\mathbb{C}}^m)$, subsequently taking the external product will also give us exactly the elements of this form. Hence $\mathrm{K}(x_0) \otimes \mathrm{K}(y_0) \to \mathrm{K}(X \times Y)$ lands isomorphically on the image of $\mathrm{K}((x_0, y_0))$ in $\mathrm{K}(X \times Y)$. Thus the lowermost map exists and is an isomorphism.

Under

$$\widetilde{\mathrm{K}}(X) \otimes \mathrm{K}(y_0) \longrightarrow \mathrm{K}(X) \otimes \mathrm{K}(Y) \xrightarrow{*} \mathrm{K}(X \times Y)$$

$\widetilde{\mathrm{K}}(X) \otimes \mathrm{K}(y_0)$ maps isomorphically onto the ideal of elements of the form $n \cdot \mathrm{pr}_X^*(a)$ for $a \in \ker(\mathrm{K}(X) \to \mathrm{K}(x_0))$ and $n \in \mathbb{N}_0$. This is also exactly the image of $\widetilde{\mathrm{K}}(X)$ in $\mathrm{K}(X \times Y)$. Hence the map third, and analogously second, from the bottom exists as in the diagram and is an isomorphism.

Now, for the uppermost of the four lower maps, let

$$
\begin{aligned}
a \in \widetilde{\mathrm{K}}(X) = \ker(\mathrm{K}(X) \to \mathrm{K}(x_0)) \subset \mathrm{K}(X) \\
b \in \widetilde{\mathrm{K}}(Y) = \ker(\mathrm{K}(Y) \to \mathrm{K}(y_0)) \subset \mathrm{K}(Y)
\end{aligned}
$$

Since the following diagram induced by inclusions (horizontal maps) and projections (vertical maps) commutes

$$
\begin{array}{ccccc}
\mathrm{K}(X) & \longrightarrow & \mathrm{K}(x_0) & = & \mathrm{K}(x_0) \\
{\scriptstyle \mathrm{pr}_X^*}\downarrow & & \downarrow & & \downarrow \\
\mathrm{K}(X \times Y) & \longrightarrow & \mathrm{K}(\{x_0\} \times Y) & \longrightarrow & \mathrm{K}(\{x_0\} \times \{y_0\})
\end{array}
$$

we have that $\mathrm{pr}_X^*(a)$ lies in $\widetilde{\mathrm{K}}(X \times Y)$ and restricts to zero in $\widetilde{\mathrm{K}}(\{x_0\} \times Y)$. Since the analogue holds for $\mathrm{pr}_Y^*(b)$, the external product $a * b = \mathrm{pr}_X^*(a)\mathrm{pr}_Y^*(b)$

lies in $\widetilde{\mathrm{K}}(X \times Y)$ and restricts to zero in both $\widetilde{\mathrm{K}}(\{x_0\} \times Y)$ and $\widetilde{\mathrm{K}}(X \times \{y_0\})$. Hence the map $\widetilde{\mathrm{K}}(X) \otimes \widetilde{\mathrm{K}}(Y) \to \widetilde{\mathrm{K}}(X \wedge Y)$ exists as claimed (here we use that the inverse of the isomorphism from Lemma 2.35 is, on the factors $\widetilde{\mathrm{K}}(\{x_0\} \times Y)$ and $\widetilde{\mathrm{K}}(X \times \{y_0\})$, actually given by the maps induced by the inclusions). This is the desired map $\star$.

Since $\widetilde{\mathrm{K}}(X) \otimes \widetilde{\mathrm{K}}(Y) \to \mathrm{K}(X) \otimes \mathrm{K}(Y)$ and the composition

$$\widetilde{\mathrm{K}}(X \wedge Y) \xrightarrow{q^*} \widetilde{\mathrm{K}}(X \times Y) \longrightarrow \mathrm{K}(X \times Y)$$

are natural, injective maps of non-unital rings, the reduced external product $\star$ is a restriction of the unreduced version $*$ (which is natural) and hence itself a natural homomorphism of non-unital rings.

From the diagram at the beginning of the proof also follows that $\star$ is an isomorphism if and only if $*$ is. $\square$

Lemma 2.37. *Let X be a pointed space and Δ the diagonal map $X \to X \wedge X$. The composition*

$$\widetilde{\mathrm{K}}(X) \otimes \widetilde{\mathrm{K}}(X) \xrightarrow{\star} \widetilde{\mathrm{K}}(X \wedge X) \xrightarrow{\Delta^*} \widetilde{\mathrm{K}}(X)$$

is equal to the normal multiplication on $\widetilde{\mathrm{K}}(X)$.

Proof. This follows from Lemma 2.25 and the fact that the diagram

$$\begin{array}{ccccc} \mathrm{K}(X) \otimes \mathrm{K}(X) & \xrightarrow{\;*\;} & \mathrm{K}(X \times X) & \xrightarrow{\Delta^*} & \mathrm{K}(X) \\ \uparrow & & \uparrow & & \uparrow \\ \widetilde{\mathrm{K}}(X) \otimes \widetilde{\mathrm{K}}(X) & \xrightarrow{\;\star\;} & \widetilde{\mathrm{K}}(X \wedge X) & \xrightarrow{\Delta^*} & \widetilde{\mathrm{K}}(X) \end{array}$$

commutes. $\square$

Recall that we denote the canonical line bundle over $\mathbb{C}\mathrm{P}^1 \cong \mathrm{S}^2$ by H. The virtual bundle $H - 1 \in \mathrm{K}(\mathrm{S}^2)$ has virtual dimension 0 over any point x_0. Thus we can consider $H-1$ to be an element of $\widetilde{\mathrm{K}}(\mathrm{S}^2)$. We can now prove the following very important result about multiplication with this element.

Theorem 2.38 (Bott Periodicity)**.** *Let X be a pointed space. Then the map*

$$\beta\colon \widetilde{\mathrm{K}}(X) \longrightarrow \widetilde{\mathrm{K}}(\mathrm{S}^2 \wedge X) \cong \widetilde{\mathrm{K}}(\Sigma^2 X), \quad a \longmapsto (H-1) \star a$$

is a natural group isomorphism.

Proof. The map β is the composition

$$\widetilde{\mathrm{K}}(X) \longrightarrow \widetilde{\mathrm{K}}(\mathrm{S}^2) \otimes \widetilde{\mathrm{K}}(X) \xrightarrow{\star} \widetilde{\mathrm{K}}(\mathrm{S}^2 \wedge X)$$

where the first map is given by $a \mapsto (H-1) \otimes a$. By Corollary 2.30, we have $\widetilde{\mathrm{K}}(\mathrm{S}^2) = \mathbb{Z}\langle H-1 \rangle$ (with trivial multiplication), hence the first map is an isomorphism of groups. By Theorem 2.29 and again Corollary 2.30, the multiplication $\mathrm{K}(\mathrm{S}^2) \otimes \mathrm{K}(X) \xrightarrow{*} \mathrm{K}(\mathrm{S}^2 \times X)$ is a ring isomorphism. Hence, by Proposition 2.36, the second map in the composition above is an isomorphism of non-unital rings. Both maps are natural, hence their composition is as well.

The natural homeomorphism $\mathrm{S}^2 \wedge X \cong \Sigma^2 X$ comes from the associativity of the smash product for compact Hausdorff spaces (see [Bro, 5.8.2]) and Corollary 1.22. □

2.10 Extending to a cohomology theory

Definition 2.39. Let (X, A) be a pair of pointed spaces with $A \subseteq X$ closed and $n \geq 0$. We set

$$\widetilde{\mathrm{K}}^{-n}(X) = \widetilde{\mathrm{K}}(\Sigma^n X) \qquad \text{and} \qquad \widetilde{\mathrm{K}}^{-n}(X, A) = \widetilde{\mathrm{K}}(\Sigma^n (X/A))$$

These are contravariant functors to abelian groups (even non-unital rings), since they are a composition of $\widetilde{\mathrm{K}}$ and Σ, i.e. for $f\colon X \to Y$ the induced map is $f^{-n} = (\Sigma^n f)^*\colon \widetilde{\mathrm{K}}(\Sigma^n Y) \to \widetilde{\mathrm{K}}(\Sigma^n X)$ and analogously for (X, A).

We write the natural long exact sequence from Proposition 2.33 in these terms

$$\begin{array}{ccccccccc}
\widetilde{\mathrm{K}}^0(X) & \longrightarrow & \widetilde{\mathrm{K}}^0(A) & & & & & & \\
\beta\downarrow\cong & & \cong\downarrow\beta & & & & & & \\
\widetilde{\mathrm{K}}^{-2}(X) & \longrightarrow & \widetilde{\mathrm{K}}^{-2}(A) & \longrightarrow & \widetilde{\mathrm{K}}^{-1}(X, A) & \longrightarrow & \widetilde{\mathrm{K}}^{-1}(X) & \longrightarrow & \widetilde{\mathrm{K}}^{-1}(A) \\
& & & \longrightarrow & \widetilde{\mathrm{K}}^0(X, A) & \longrightarrow & \widetilde{\mathrm{K}}^0(X) & \longrightarrow & \widetilde{\mathrm{K}}^0(A)
\end{array}$$

The occurring square commutes by the naturality of β.

Definition 2.40. Let X be a pointed space. For $k \geq 0$ we set

$$\widetilde{\mathrm{K}}^{2k}(X) = \widetilde{\mathrm{K}}^0(X) \qquad \text{and} \qquad \widetilde{\mathrm{K}}^{2k+1}(X) = \widetilde{\mathrm{K}}^{-1}(X)$$

and analogously for pairs.

By definition, these are contravariant functors to abelian groups, i.e. for $f\colon X \to Y$ we set $f^{2k} = f^0$ and $f^{2k+1} = f^{-1}$.

With this definition, the sequence above extends to the right to form the standard, infinite in both directions, natural long exact sequence of a pair (for the naturality we use again that β is natural). Since it is six-periodic, we obtain the natural long exact circle (the diagram is not meant to be commutative)

$$\begin{array}{ccccc}
\widetilde{\mathrm{K}}^0(X, A) & \longrightarrow & \widetilde{\mathrm{K}}^0(X) & \longrightarrow & \widetilde{\mathrm{K}}^0(A) \\
\uparrow & & & & \downarrow \\
\widetilde{\mathrm{K}}^1(A) & \longleftarrow & \widetilde{\mathrm{K}}^1(X) & \longleftarrow & \widetilde{\mathrm{K}}^1(X, A)
\end{array}$$

Definition 2.41. We set

$$\widetilde{\mathrm{K}}^*(X) = \widetilde{\mathrm{K}}^0(X) \oplus \widetilde{\mathrm{K}}^1(X) \qquad \text{and} \qquad \widetilde{\mathrm{K}}^*(X,A) = \widetilde{\mathrm{K}}^0(X,A) \oplus \widetilde{\mathrm{K}}^1(X,A)$$

These are contravariant functors to $\mathbb{Z}/2$-graded abelian groups by setting, for $f\colon X \to Y$, $f^* = f^0 \oplus f^1$.

As for singular cohomology, we can define a product on the graded groups $\widetilde{\mathrm{K}}^*(X)$.

Proposition 2.42. *There is a natural multiplication*

$$\widetilde{\mathrm{K}}^*(X) \otimes \widetilde{\mathrm{K}}^*(X) \longrightarrow \widetilde{\mathrm{K}}^*(X)$$

restricting to the usual multiplication on $\widetilde{\mathrm{K}}^0(X)$. *This makes* $\widetilde{\mathrm{K}}^*(X)$ *into a* $\mathbb{Z}/2$*-graded non-unital ring which is graded commutative, i.e.* $ab = (-1)^{nm}ba$ *for* $a \in \widetilde{\mathrm{K}}^n(X)$ *and* $b \in \widetilde{\mathrm{K}}^m(X)$.

Proof. We define a natural external product

$$\star\colon \widetilde{\mathrm{K}}^n(X) \otimes \widetilde{\mathrm{K}}^m(Y) \longrightarrow \widetilde{\mathrm{K}}^{n+m}(X \wedge Y)$$

via the composition

$$\widetilde{\mathrm{K}}(\mathrm{S}^n \wedge X) \otimes \widetilde{\mathrm{K}}(\mathrm{S}^m \wedge Y) \xrightarrow{\star} \widetilde{\mathrm{K}}(\mathrm{S}^n \wedge X \wedge \mathrm{S}^m \wedge Y) \cong \widetilde{\mathrm{K}}(\mathrm{S}^{n+m} \wedge X \wedge Y)$$

Here we use that the smash product is commutative and, for compact Hausdorff spaces, associative up to the canonical homeomorphisms (see [Bro, 5.8.2] for the associativity). The identification $\mathrm{S}^n \wedge \mathrm{S}^m \cong \mathrm{S}^{n+m}$ comes from

$$\mathrm{S}^n \wedge \mathrm{S}^m = (\mathbb{R}^n \cup \{\infty\}) \wedge (\mathbb{R}^m \cup \{\infty\}) \xrightarrow{\cong} (\mathbb{R}^{n+m} \cup \{\infty\}) = \mathrm{S}^{n+m}$$

given by $[(x,y)] \mapsto (x,y)$ for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $[(\infty,\infty)] \mapsto \infty$, which is a homeomorphism by Lemma 1.21. This definition of $\star$ restricts to our previous instance of $\star$ for $n = m = 0$.

From this and the isomorphism β, we obtain a multiplication on $\widetilde{\mathrm{K}}^*(X)$ via

$$\widetilde{\mathrm{K}}^n(X) \otimes \widetilde{\mathrm{K}}^m(X) \xrightarrow{\star} \widetilde{\mathrm{K}}^{n+m}(X \wedge X) \xrightarrow{\Delta^*} \widetilde{\mathrm{K}}^{n+m}(X)$$

where $\Delta\colon X \to X \wedge X$ is the diagonal map. By Lemma 2.37, this extends the normal multiplication of $\widetilde{\mathrm{K}}^0(X) = \widetilde{\mathrm{K}}(X)$.

For the graded commutativity consider the diagram

$$\begin{array}{ccc}
\widetilde{\mathrm{K}}(\mathrm{S}^n \wedge X) \otimes \widetilde{\mathrm{K}}(\mathrm{S}^m \wedge X) & \longrightarrow & \widetilde{\mathrm{K}}(\mathrm{S}^m \wedge X) \otimes \widetilde{\mathrm{K}}(\mathrm{S}^n \wedge X) \\
\star\downarrow & & \downarrow\star \\
\widetilde{\mathrm{K}}((\mathrm{S}^n \wedge X) \wedge (\mathrm{S}^m \wedge X)) & \xrightarrow{\tau^*} & \widetilde{\mathrm{K}}((\mathrm{S}^m \wedge X) \wedge (\mathrm{S}^n \wedge X)) \\
\cong\downarrow & & \downarrow\cong \\
\widetilde{\mathrm{K}}((\mathrm{S}^n \wedge \mathrm{S}^m) \wedge (X \wedge X)) & \xrightarrow{(\tau\wedge\tau)^*} & \widetilde{\mathrm{K}}((\mathrm{S}^m \wedge \mathrm{S}^n) \wedge (X \wedge X)) \\
(\mathrm{id}\wedge\Delta)^*\downarrow & & \downarrow(\mathrm{id}\wedge\Delta)^* \\
\widetilde{\mathrm{K}}((\mathrm{S}^n \wedge \mathrm{S}^m) \wedge X) & \xrightarrow{(\tau\wedge\mathrm{id})^*} & \widetilde{\mathrm{K}}((\mathrm{S}^m \wedge \mathrm{S}^n) \wedge X) \\
\cong\uparrow & & \uparrow\cong \\
\widetilde{\mathrm{K}}(\mathrm{S}^{n+m} \wedge X) & & \widetilde{\mathrm{K}}(\mathrm{S}^{n+m} \wedge X)
\end{array}$$

where the uppermost horizontal map is given by switching the two factors of the tensor product and $\tau\colon Y \wedge Z \to Z \wedge Y$ is always the map exchanging the two corresponding factors.

The lower two squares commute since they already do so on the level of spaces. For the uppermost square consider the corresponding unreduced diagram (in the general case)

$$\begin{array}{ccc}
\mathrm{K}(Y) \otimes \mathrm{K}(Z) & \longrightarrow & \mathrm{K}(Z) \otimes \mathrm{K}(Y) \\
*\downarrow & & \downarrow * \\
\mathrm{K}(Y \times Z) & \xrightarrow{\tau^*} & \mathrm{K}(Z \times Y)
\end{array}$$

It commutes since

$$\tau^*\Big(\big(\mathrm{pr}_Y^{Y\times Z}\big)^*(a) \cdot \big(\mathrm{pr}_Z^{Y\times Z}\big)^*(b)\Big) = \big(\mathrm{pr}_Z^{Z\times Y}\big)^*(b) \cdot \big(\mathrm{pr}_Y^{Z\times Y}\big)^*(a)$$

as $\mathrm{pr}_i^{Y\times Z} \circ \tau = \mathrm{pr}_i^{Z\times Y}$ for $i \in \{Y, Z\}$. Since the reduced square is just a restriction of this one, it also commutes.

It remains to show that the map $\widetilde{\mathrm{K}}(\mathrm{S}^{n+m} \wedge X) \to \widetilde{\mathrm{K}}(\mathrm{S}^{n+m} \wedge X)$ given by multiplication with $(-1)^{nm}$ makes the large diagram commute. This holds when $n = 0$ or $m = 0$ since, in this case, the diagram

$$\begin{array}{ccc}
\mathrm{S}^n \wedge \mathrm{S}^m & \xrightarrow{\tau} & \mathrm{S}^m \wedge \mathrm{S}^n \\
\downarrow & & \downarrow \\
\mathrm{S}^{n+m} & \xrightarrow{\mathrm{id}} & \mathrm{S}^{n+m}
\end{array}$$

commutes. If $n = m = 1$, the map $\mathrm{S}^n \wedge \mathrm{S}^m \to \mathrm{S}^m \wedge \mathrm{S}^n$ switching the two factors corresponds to exchanging the two coordinates of $\mathbb{R}^2$ in $\mathrm{S}^2 = \mathbb{R}^2 \cup \{*\}$, i.e. reflecting across an equator of S^2. The next lemma finishes the proof. $\square$

Lemma 2.43. *Let $r\colon S^2 \to S^2$ be the reflection across an equator. Then the map $(r \wedge \mathrm{id})^*\colon \widetilde{K}(S^2 \wedge X) \to \widetilde{K}(S^2 \wedge X)$ is given by multiplication with -1.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{K}(S^2) \otimes \widetilde{K}(X) & \xrightarrow{r^* \otimes \mathrm{id}} & \widetilde{K}(S^2) \otimes \widetilde{K}(X) \\ \star\big\downarrow & & \big\downarrow\star \\ \widetilde{K}(S^2 \wedge X) & \xrightarrow{(r\wedge \mathrm{id})^*} & \widetilde{K}(S^2 \wedge X) \end{array}$$

The vertical maps are isomorphisms by Bott Periodicity. Hence it is enough to show that $r^*\colon \widetilde{K}(S^2) \to \widetilde{K}(S^2)$ is multiplication with -1. This is equivalent to $[E] + [r^*E] = [\underline{\mathbb{C}}^0]$, i.e. $E \oplus r^*E \sim \underline{\mathbb{C}}^0$, for all vector bundles E over S^2.

As in the proof of Lemma 2.27, we consider a clutching function $f\colon S^1 \to \mathrm{GL}_n(\mathbb{C})$ corresponding to E. Choosing the S^1 to be the equator along which r reflects, we see that r^*E corresponds to the clutching function $i \circ f$, where $i\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is given by $A \mapsto A^{-1}$.

Now, by Lemma 2.28, we have

$$f \oplus (i \circ f) \simeq (f \cdot (i \circ f)) \oplus \mathrm{const}_{\mathbb{1}_n} = \mathrm{const}_{\mathbb{1}_n} \oplus \mathrm{const}_{\mathbb{1}_n} = \mathrm{const}_{\mathbb{1}_{2n}}$$

and hence $E \oplus r^*E \cong \underline{\mathbb{C}}^{2n}$. □

Now, at the end of this section, we do the following computation which we will need later.

Lemma 2.44. *If X is a pointed finite CW-complex with only even-dimensional cells, then we have $\widetilde{K}^{2n+1}(X) \cong 0$ for any $n \in \mathbb{Z}$.*

Proof. We prove this by induction on the number of cells. The statement is clear for $X = \mathrm{pt}$, as $\widetilde{K}^{2n+1}(\mathrm{pt}) = \widetilde{K}(S^1) \cong 0$, by Lemma 2.5.

Now let $X = X' \cup_f D^{2n}$. From the long exact sequence of the pair (X, X') we obtain the following exact sequence

$$\widetilde{K}^{2n+1}(X, X') \longrightarrow \widetilde{K}^{2n+1}(X) \longrightarrow \widetilde{K}^{2n+1}(X')$$

The group on the right is trivial by the induction hypothesis. Since, by Bott Periodicity and again Lemma 2.5, we also have

$$\widetilde{K}^{2n+1}(X, X') = \widetilde{K}(\Sigma(X/X')) \cong \widetilde{K}(S^{2n+1}) \cong \widetilde{K}(S^1) \cong 0$$

the middle term is trivial as well. □

2.11 Outlook

To see that the theory we developed over the last section is actually useful, we state here some of the classical main applications. The first one is the Hopf Invariant One Problem.

Theorem 2.45. *There exists a map* $S^{4n-1} \to S^{2n}$ *only if* $n \in \{1, 2, 4\}$*.*

This is [HaK, Theorem 2.19]. It is proven by endowing the rings $K(X)$ with even more structure, coming from exterior powers of vector bundles. From this one derives some algebraic relations which, after applying a little bit of simple number theory, yield the stated result.

From this one can then deduce the following.

Theorem 2.46. *The following two statements are only true for* $n \in \{1, 2, 4, 8\}$*:*

1. $\mathbb{R}^n$ *is a division algebra.*
2. S^{n-1} *is parallelizable, i.e. its tangent bundle is trivial.*

This is [HaK, Theorem 2.16].

3 Foundations from stable homotopy theory

In the next section we will need some language from stable homotopy theory, mainly spectra and their associated (co)homology theories, but also related concepts. This is what this section focuses on. We will mostly follow the book by Switzer [Sw] and, in the beginning, the book by Adams [Ad] (the main difference being that Switzer does not introduce sequential spectra). For a newer work treating this, see the book by Rudyak [Ru], though he omits quite a few proofs, referencing Switzer.

Note that we will prove hardly anything in this section, instead focusing on introducing the needed concepts. For the most relevant statements references are given, but in most definitions and side remarks there will also be things to check. For those we refer to the books named above.

3.1 Sequential spectra

Definition 3.1. A *sequential spectrum* is a sequence of pointed spaces $E = (E_n)_{n\in\mathbb{Z}}$ (the *component spaces*) together with pointed maps $\epsilon_n \colon \Sigma E_n \to E_{n+1}$ for $n \in \mathbb{Z}$ (the *structure maps*).

A *map of sequential spectra* $f \colon E \to E'$ is a sequence of pointed maps $(f_n \colon E_n \to E'_n)_{n\in\mathbb{Z}}$ such that they commute with the structure maps, i.e. we have the following commuting diagram for each $n \in \mathbb{Z}$

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\epsilon_n} & X_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma E'_n & \xrightarrow{\epsilon'_n} & Y_{n+1} \end{array}$$

We denote the category of sequential spectra with maps of sequential spectra by **SeqSpec**.

Definition 3.2. An Ω-*spectrum* is a sequential spectrum E such the the maps $E_n \to \Omega E_{n+1}$, given as the adjuncts of the structure maps σ_n under the adjunction of reduced suspension and loop space, are weak homotopy equivalences.

Definition 3.3. There is a covariant functor $\Sigma^\infty\colon \mathbf{Top}_* \to \mathbf{SeqSpec}$ from pointed spaces to sequential spectra given on objects by

$$(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & n \geq 0 \\ \mathrm{pt} & n < 0 \end{cases}$$
$$\epsilon_n = \begin{cases} \mathrm{id}\colon \Sigma(\Sigma^n X) \to \Sigma^{n+1} X & n \geq 0 \\ \mathrm{const}\colon \Sigma\mathrm{pt} \cong \mathrm{pt} \to (\Sigma^\infty X)_{n+1} & n < 0 \end{cases}$$

and on functions $f\colon X \to Y$ by

$$(\Sigma^\infty f)_n = \Sigma^n f\colon \Sigma^n X \to \Sigma^n Y$$

We call $\Sigma^\infty X$ the *suspension spectrum* of X.

Definition 3.4. We set $\mathbb{S} = \Sigma^\infty \mathrm{S}^0$, the *sphere spectrum*. Hence $\mathbb{S}_n \cong \mathrm{S}^n$ for $n \geq 0$ and $\mathbb{S}_n \cong \mathrm{pt}$ for $n < 0$.

Definition 3.5. For a sequential spectrum E and $k \in \mathbb{Z}$ define the sequential spectrum $\Sigma^k E$ by $(\Sigma^k E)_n = E_{n+k}$ with structure maps also shifted by k. This gives us a functor $\Sigma^k\colon \mathbf{SeqSpec} \to \mathbf{SeqSpec}$ for each $k \in \mathbb{Z}$. If $k = 1$, we write just $\Sigma = \Sigma^1$.

The functor $\Sigma\colon \mathbf{SeqSpec} \to \mathbf{SeqSpec}$ is invertible, with inverse given by Σ^{-1}.

3.2 CW-spectra

Definition 3.6. A *CW-spectrum* is a sequential spectrum E such that each component space E_n is a pointed CW-complex and such that the structure maps $\epsilon_n\colon \Sigma E_n \to E_{n+1}$ are inclusions of subcomplexes.

For a pointed CW-complex X its suspension spectrum $\Sigma^\infty X$ is a CW-spectrum, in particular $\mathbb{S} = \Sigma^\infty \mathrm{S}^0$. Similarly, if E is a CW-spectrum, then the $\Sigma^k E$ are again CW-spectra.

From now on we will only work with CW-spectra. Since not all naturally arising spectra are CW-spectra, we need the following result. It tells us that we are not significantly limiting ourselves when restricting to CW-spectra.

Proposition 3.7. *Let E be a sequential spectrum. Then there is a CW-spectrum E' and a map of sequential spectra $f\colon E' \to E$ such that $f_n\colon E'_n \to E_n$ is a homotopy equivalence for all $n \in \mathbb{Z}$.*

This statement can be found in [Ru, II, Lemma-Definition 1.19].

Definition 3.8. For a CW-spectrum E and a pointed CW-complex X, define the CW-spectrum $E \wedge X$ by $(E \wedge X)_n = E_n \wedge X$ (we have to be a little careful with the topology, see Remark 1.7) with structure maps

$$\Sigma(E_n \wedge X) = (\Sigma E_n) \wedge X \xrightarrow{\epsilon_n \wedge \mathrm{id}} E_{n+1} \wedge X$$

We obtain a functor $\wedge\colon \mathbf{CWSpec} \times \mathbf{CW}_* \to \mathbf{CWSpec}$ by sending $(f\colon E \to F, g\colon X \to Y)$ to $f_n \wedge g\colon E_n \wedge X \to F_n \wedge Y$.

We have, for example, that $\mathbb{S} \wedge X \cong \Sigma^\infty X$.

Definition 3.9. For a family of CW-spectra $(E_i)_{i \in I}$ define their *wedge sum* $\bigvee_{i \in I} E_i$ as having n-th component $\bigvee_{i \in I}(E_i)_n$ and structure maps

$$\Sigma \bigvee_{i \in I}(E_i)_n \cong \bigvee_{i \in I} \Sigma(E_i)_n \xrightarrow{\bigvee_{i \in I} \epsilon_n^{E_i}} \bigvee_{i \in I}(E_i)_{n+1}$$

We have that $\Sigma^\infty X \vee \Sigma^\infty Y \cong \Sigma^\infty(X \vee Y)$, since there is a homeomorphism $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

Definition 3.10. A *CW-subspectrum* of a CW-spectrum E is a CW-spectrum E' together with a map of sequential spectra $\iota\colon E' \to E$ such that $\iota_n\colon E'_n \to E_n$ is the inclusion of a subcomplex for all $n \in \mathbb{Z}$.

A CW-subspectrum $E' \subseteq E$ is *cofinal* if, for all $n \in \mathbb{Z}$ and each cell of E_n, some E'_{n+k} contains a suspension of that cell.

Definition 3.11. Let E and F be two CW-spectra. We say that two maps of sequential spectra $f'\colon E' \to F$ and $f''\colon E'' \to F$, where $E', E'' \subseteq E$ are cofinal CW-subspectra, are equivalent if they agree on $E' \cap E''$ (which is again a cofinal CW-subspectrum).

A *CW-map* is an equivalence class of such maps. We write **CWSpec** for the category of CW-spectra with CW-maps.

One has to be a little careful when defining composition, in that the image of a representative of the first map does not necessarily have to lie in the cofinal subspectrum on which a representative of the second map is defined. But this can be fixed by restricting to a smaller cofinal subspectrum in the domain of the first map.

Definition 3.12. A *homotopy* between two CW-maps $f_0, f_1\colon E \to F$ is a CW-map $E \wedge [0,1]_+ \to F$ such that $F \circ \iota_0 = f_0$ and $F \circ \iota_1 = f_1$, where $\iota_0\colon E \wedge \{0\}_+ \to E \wedge [0,1]_+$ and $\iota_1\colon E \wedge \{1\}_+ \to E \wedge [0,1]_+$ are induced by the inclusions.

We write **HoSpec** for the category of CW-spectra with CW-maps up to homotopy. We have a functor $\mathbf{CWSpec} \to \mathbf{HoSpec}$ sending a CW-spectrum to itself and a CW-map to its equivalence class up to homotopy.

By composing with the functor to **HoSpec**, we can consider the suspension spectrum also as a functor $\Sigma^\infty\colon \mathbf{CW}_* \to \mathbf{HoSpec}$.

The functors $\wedge$ and Σ^k descend to functors $\wedge\colon \mathbf{HoSpec} \times \mathbf{CW}_* \to \mathbf{HoSpec}$ and $\Sigma\colon \mathbf{HoSpec} \to \mathbf{HoSpec}$, i.e. if E and F are CW-spectra, X is a pointed CW-complex, and two CW-maps $E \to F$ are homotopic, then the induced maps $E \wedge X \to F \wedge X$ respectively $\Sigma^k X \to \Sigma^k Y$ are again homotopic.

Proposition 3.13. *The wedge sum of CW-spectra constitutes the coproduct in the category* **HoSpec**.

This can be found in [Sw, Proposition 8.18].

Definition 3.14. For E and F two CW-spectra, we write $[E, F]$ for the set of CW-maps $E \to F$ up to homotopy.

Proposition 3.15. *There is a natural bijection*

$$[E, F] \cong [E \wedge \mathrm{S}^1, F \wedge \mathrm{S}^1]$$

where $E, F \in \mathbf{CWSpec}$.

From this we get, via the pinching map $\mathrm{S}^2 \to \mathrm{S}^2 \vee \mathrm{S}^2$*, an abelian group structure on* $[E, F] \cong [E \wedge \mathrm{S}^2, F \wedge \mathrm{S}^2]$ *such that composition is bilinear.*

This is [Sw, Corollary 8.27].

Definition 3.16. For a CW-spectrum E we set $\pi_n E = [\Sigma^n \mathbb{S}, E]$, the *stable homotopy groups* of E. We obtain a functor $\pi_*\colon \mathbf{CWSpec} \to \mathbf{Ab}^{\mathbb{Z}}$ from CW-spectra to graded abelian groups.

Remark 3.17. Another, often used, definition is to set $\pi_n E = \operatorname{colim}_k \pi_{n+k}(E_k)$ where the colimit runs over maps

$$\pi_{n+k}(E_k) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma E_k) \xrightarrow{(\epsilon_k)_*} \pi_{n+k+1}(E_{k+1})$$

This is equivalent to the one we stated (cf. [Sw, 8.21]).

The following two lemmas will be useful later.

Lemma 3.18. *For* $E \in \mathbf{CWSpec}$ *there is a natural homotopy equivalence* $\Sigma E \simeq E \wedge \mathrm{S}^1$.

This is [Sw, Theorem 8.26].

Lemma 3.19. *Let* X *be a pointed CW-complex. Then there is a homotopy equivalence* $\Sigma^\infty X_+ \simeq \Sigma^\infty(X \vee \mathrm{S}^0) \cong \Sigma^\infty X \vee \mathbb{S}$*. Here* X_+ *and* $X \vee \mathrm{S}^0$ *are homeomorphic CW-complexes, but the first has as basepoint the extra point* $+$ *and the second the basepoint of* X.

Proof. Both $\Sigma(X_+)$ and $\Sigma(X \vee \mathrm{S}^0)$ are quotients of the unreduced suspension $\mathrm{S}X_+$ (which does not depend on the basepoint). By Lemma 1.12, we have that both quotient maps are homotopy equivalences, as X_+ and $X \vee \mathrm{S}^0$ are CW-complexes and hence well-pointed. Thus $\Sigma(X_+) \simeq \Sigma(X \vee \mathrm{S}^0)$ as spaces and, by Lemma 1.14, they are also homotopy equivalent relative to the basepoints. □

3.3 Generalized (co)homology

Definition 3.20. A *generalized reduced homology theory* is a covariant, homotopy invariant functor $\tilde{h}_*\colon \mathbf{CW}_* \to \mathbf{Ab}^{\mathbb{Z}}$ from pointed CW-complexes to ($\mathbb{Z}$-)graded abelian groups together with a sequence of natural isomorphisms $\sigma_n\colon \tilde{h}_n \Rightarrow \tilde{h}_{n+1} \circ \Sigma$ for $n \in \mathbb{Z}$ (the *suspension isomorphisms*) such that for inclusions of subcomplexes of pointed CW-complexes $\iota\colon A \to X$ the following induced sequence is exact

$$\tilde{h}_* A \xrightarrow{\iota_*} \tilde{h}_* X \xrightarrow{q_*} \tilde{h}_*(X/A)$$

where $q\colon X \to X/A$ is the quotient map.

A generalized reduced homology theory is said to be *additive* if the canonical map $\bigoplus_{i\in I} \tilde{h}_* X_i \to \tilde{h}_*\big(\bigvee_{i\in I} X_i\big)$, induced by the inclusions $X_i \to \bigvee_{i\in I} X_i$, is an isomorphism for all families of pointed CW-complexes $(X_i)_{i\in I}$.

A *map of generalized reduced homology theories* is a natural transformation $\eta\colon \tilde{h}_* \Rightarrow \tilde{g}_*$ such that it commutes with the suspension isomorphisms.

Definition 3.21. We can associate to a CW-spectrum E a generalized reduced homology theory $\widetilde{E}_*$ by setting $\widetilde{E}_n X = \pi_n(E \wedge X) = [\Sigma^n \mathbb{S}, E \wedge X]$ and

$$\sigma_n\colon [\Sigma^n \mathbb{S}, E \wedge X] \cong [\Sigma\Sigma^n \mathbb{S}, \Sigma E \wedge X] \cong [\Sigma^{n+1}\mathbb{S}, E \wedge \Sigma X]$$

where the first isomorphism comes from invertibility of Σ and the second from Lemma 3.18. The functoriality is given by sending $f\colon X \to Y$ to composition with the map $\mathrm{id} \wedge f\colon E \wedge X \to E \wedge Y$.

Generalized reduced homology theories obtained in this way are additive (cf. [Sw, Corollary 8.36]).

This construction is functorial in $E \in \mathbf{CWSpec}$. A CW-map $f\colon E \to F$ induces a map of generalized reduced homology theories $\widetilde{E}_* \Rightarrow \widetilde{F}_*$ by composing with $f \wedge \mathrm{id}$.

Definition 3.22. For $E = \mathbb{S}$ we obtain $\widetilde{\mathbb{S}}_* X = \pi_*(\Sigma^\infty X)$, i.e. taking the stable homotopy groups of the suspension spectrum forms a generalized reduced homology theory. We will denote it by $\pi^{\mathrm{s}}_* X$ and call $\pi^{\mathrm{s}}_n X$ the n-th *stable homotopy group* of X.

A homotopy equivalence of CW-spectra induces an isomorphism of generalized reduced homology theories, hence this construction is also functorial in $E \in \mathbf{HoSpec}$.

Definition 3.23. A *generalized reduced cohomology theory* is a contravariant, homotopy invariant functor $\tilde{h}^*\colon \mathbf{CW}_*^{\mathbf{op}} \to \mathbf{Ab}^{\mathbb{Z}}$ from pointed CW-complexes to ($\mathbb{Z}$-)graded abelian groups together with a sequence of natural isomorphisms $\sigma^n\colon \tilde{h}^{n+1} \circ \Sigma \Rightarrow \tilde{h}^n$ for $n \in \mathbb{Z}$ (the *suspension isomorphisms*) such that for inclusions of subcomplexes of pointed CW-complexes $\iota\colon A \to X$ the following induced sequence is exact

$$\tilde{h}^*(X/A) \xrightarrow{q^*} \tilde{h}^* X \xrightarrow{\iota^*} \tilde{h}^* A$$

where $q\colon X \to X/A$ is the quotient map.

A generalized reduced cohomology theory is said to be *additive* if the canonical map $\tilde{h}^*\big(\bigvee_{i\in I} X_i\big) \to \prod_{i\in I} \tilde{h}^* X_i$, induced by the inclusions $X_i \to \bigvee_{i\in I} X_i$, is an isomorphism for all families of pointed CW-complexes $(X_i)_{i\in I}$.

A *map of generalized reduced cohomology theories* is a natural transformation $\eta\colon \tilde{h}^* \Rightarrow \tilde{g}^*$ such that it commutes with the suspension isomorphisms.

Definition 3.24. A CW-spectrum E represents a generalized reduced cohomology theory $\widetilde{E}^*$ by setting $\widetilde{E}^n X = [\Sigma^\infty X, \Sigma^n E]$ and

$$\sigma^n\colon [\Sigma^\infty \Sigma X, \Sigma^{n+1} E] \cong [\Sigma\Sigma^\infty X, \Sigma^{n+1} E] \cong [\Sigma^\infty X, \Sigma^n E]$$

using that $\Sigma^\infty \Sigma X$ is cofinal in $\Sigma\Sigma^\infty X$ and the invertibility of Σ. The functoriality is given by sending $f\colon X \to Y$ to precomposition with $\Sigma^\infty f\colon \Sigma^\infty X \to \Sigma^\infty Y$.

Generalized reduced cohomology theories obtained in this way are additive (cf. [Sw, p. 146]).

This construction is functorial in $E \in$ **CWSpec**. A CW-map $f\colon E \to F$ gives us a map of generalized reduced cohomology theories $\widetilde{E}^* \Rightarrow \widetilde{F}^*$ by composing with $\Sigma^n f$.

Similarly to homology, we have that a homotopy equivalence of CW-spectra induces an isomorphism of generalized reduced cohomology theories, hence we have a construction also functorial in $E \in$ **HoSpec**.

If E is an Ω-spectrum, there is a natural (in X) bijection $\widetilde{E}^n X \cong [X, E_n]$ (cf. [Sw, Theorem 8.42]).

The following is one of the main motivations for studying spectra. It gives us an equivalence between Ω-spectra and cohomology theories.

Theorem 3.25 (Brown Representability). *Every additive generalized reduced cohomology theory $\tilde{h}^*$ is, up to isomorphism, represented by an Ω-spectrum E.*

A proof can be found in [Sw, Theorem 9.27].

3.4 Monoidal categories and the smash product of spectra

We will now introduce some general categorical notions that will help us to formulate statement we will need further down the line.

Definition 3.26. A *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, an object $1 \in \mathcal{C}$ and three natural isomorphisms corresponding to associativity of $\otimes$ as well as left respectively right unitality of 1. Furthermore these natural isomorphisms have to fulfill certain coherence conditions, i.e. a number of diagrams built from them one would want to commute actually do commute. We will not state them here. They can be found in [Ma, XI.1].

A monoidal category is *symmetric* if there is additionally a natural isomorphism corresponding to commutativity of $\otimes$ such that a further three diagrams commute. They can again be found in [Ma, XI.1].

Examples include $(\mathbf{hCW}, \times, \mathrm{pt})$ and $(\mathbf{hCW}_*, \wedge, \mathrm{S}^0)$ which will appear again later. They are both even symmetric monoidal categories.

Definition 3.27. A *monoid* in a monoidal category $\mathcal{C}$ is an object $M \in \mathcal{C}$ together with two morphisms $\mu\colon M \otimes M \to M$ and $i\colon 1 \to M$ such that the two diagrams

$$\begin{array}{ccccc} M \otimes (M \otimes M) & & \xleftrightarrow{\;\cong\;} & & (M \otimes M) \otimes M \\ {\scriptstyle \mathrm{id}\otimes\mu}\big\downarrow & & & & \big\downarrow{\scriptstyle \mu\otimes\mathrm{id}} \\ M \otimes M & \xrightarrow{\;\mu\;} & M & \xleftarrow{\;\mu\;} & M \otimes M \end{array}$$

and

$$\begin{array}{ccccc} 1 \otimes M & \xrightarrow{\;i\otimes\mathrm{id}\;} & M \otimes M & \xleftarrow{\;\mathrm{id}\otimes i\;} & M \otimes 1 \\ & {\scriptstyle\cong}\nwarrow\!\searrow & \big\downarrow{\scriptstyle \mu} & \swarrow\!\nearrow{\scriptstyle\cong} & \\ & & M & & \end{array}$$

commute.

A monoid M in a symmetric monoidal category is *commutative* if in addition the following diagram commutes

$$\begin{array}{ccc} M \otimes M & \underset{c}{\xleftrightarrow{\;\cong\;}} & M \otimes M \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & M & \end{array}$$

where c is the natural isomorphism corresponding to commutativity of $\otimes$.

A *map* between two monoids M and N in a monoidal category $\mathcal{C}$ is a morphism $f\colon M \to M'$ such that the following two diagrams commute

$$\begin{array}{ccc} M \otimes M & \xrightarrow{\;\mu_M\;} & M \\ {\scriptstyle f\otimes f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ N \otimes N & \xrightarrow{\;\mu_N\;} & N \end{array} \qquad\qquad \begin{array}{ccc} & \mathbb{S} & \\ {\scriptstyle i_M}\swarrow & & \searrow{\scriptstyle i_N} \\ M & \xrightarrow{\;\;f\;\;} & N \end{array}$$

We obtain, for every monoidal category $\mathcal{C}$, a category $\mathbf{Mon}_{\mathcal{C}}$ with objects the monoids in $\mathcal{C}$ and morphisms given by maps of monoids.

Definition 3.28. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two monoidal categories. A *(lax) monoidal functor* between $\mathcal{C}$ and $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\phi\colon FA \otimes_{\mathcal{D}} FB \to F(A \otimes_{\mathcal{C}} B)$ (of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$) and a morphism $\epsilon\colon 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ such that ϕ and ϵ are compatible with the natural isomorphisms coming with $\mathcal{C}$ and $\mathcal{D}$. For a precise statement regarding the diagrams that have to commute we again refer to [Ma, XI.2].

A *strong monoidal functor* is a lax monoidal functor such that the natural transformation ϕ and the morphism ϵ are isomorphisms.

A monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor $\mathbf{Mon}_{\mathcal{C}} \to \mathbf{Mon}_{\mathcal{D}}$ which we denote by $\mathbf{Mon}(F)$.

An example for a strong monoidal functor is $(-)_+ \colon \mathbf{hCW} \to \mathbf{hCW}_*$.

The following statement will later be useful.

Proposition 3.29. *Let $L \colon \mathcal{C} \to \mathcal{D}$ and $R \colon \mathcal{D} \to \mathcal{C}$ be two functors between two monoidal categories. Also let $L \dashv R$ be an adjunction. If L is strong monoidal, then it induces the structure of a monoidal functor on R and the adjunction lifts to an adjunction $\mathbf{Mon}(L) \dashv \mathbf{Mon}(R)$ of functors between the categories $\mathbf{Mon}_{\mathcal{C}}$ and $\mathbf{Mon}_{\mathcal{D}}$.*

This is a special case of so called doctrinal adjunction [Ke].

We can now formulate the following proposition, which is one of the most important facts about spectra.

Proposition 3.30. *The category* **HoSpec** *is a* symmetric monoidal category*, i.e. there is a functor $\wedge \colon \mathbf{HoSpec} \times \mathbf{HoSpec} \to \mathbf{HoSpec}$ satisfying the conditions from Definition 3.26. The unit element is the sphere spectrum $\mathbb{S}$. Moreover we have*

1. *For $X \in \mathbf{CW}_*$ and $E \in \mathbf{HoSpec}$ there is a natural isomorphism*

$$E \wedge X \simeq E \wedge \Sigma^\infty X$$

   In particular, this gives us a natural isomorphism

$$\Sigma^\infty (X \wedge Y) \cong (\Sigma^\infty X) \wedge Y \simeq \Sigma^\infty X \wedge \Sigma^\infty Y$$

   Hence Σ^∞ is a strong monoidal functor.

2. *For $E, F \in \mathbf{HoSpec}$ there are natural isomorphisms*

$$\Sigma E \wedge F \simeq \Sigma(E \wedge F) \simeq E \wedge \Sigma F$$

The construction can be found in [Ad, III, Theorem 4.1] or [Sw, p. 254ff. and Theorem 13.40]. Properties 1 and 2 are [Sw, Corollary 13.39] and [Sw, Proposition 13.46] respectively.

This monoidal structure will allow us the definitions and statements in the following subsection. But first we state the following proposition, which gives us a right adjoint to the functor Σ^∞.

Proposition 3.31. *There is a functor $\Omega^\infty \colon \mathbf{HoSpec} \to \mathbf{hCW}_*$ such that there is, for $X \in \mathbf{hCW}_*$ and $E \in \mathbf{HoSpec}$, a natural bijection $[\Sigma^\infty X, E] \cong [X, \Omega^\infty E]_*$, where $[-,-]_*$ denotes the set of equivalence classes of pointed maps up to basepoint preserving homotopy. In other words, there is an adjunction $\Sigma^\infty \dashv \Omega^\infty$.*

Furthermore, if E is an Ω-spectrum, we have $\Omega^\infty E \simeq E_0$, i.e. it is just given by taking the zeroth component.

This is almost exactly [Ru, II, Proposition-Definition 3.27 and Corollary 3.29], though we need to combine it with the fact that homotopy equivalences of CW-spectra induce isomorphisms of cohomology theories.

3.5 Ring spectra and products

We want to be able to define products on (co)homology theories associated to a spectrum. For this we need the following notion.

Definition 3.32. A *ring spectrum* is a monoid in the symmetric monoidal category (**HoSpec**, $\wedge$). It is *commutative* if it is a commutative monoid.

Proposition 3.33. *Let $E \in$ **HoSpec** be a ring spectrum with multiplication $\mu\colon E \wedge E \to E$ and identity $i\colon \mathbb{S} \to E$.*

1. *We get a natural associative bilinear product*

$$\wedge\colon \widetilde{E}_n X \otimes \widetilde{E}_m Y \longrightarrow \widetilde{E}_{n+m}(X \wedge Y)$$

by setting, for $f\colon \Sigma^n\mathbb{S} \to E \wedge X$ and $g\colon \Sigma^m\mathbb{S} \to E \wedge X$,

$$\begin{aligned} f \wedge g\colon \Sigma^{n+m}\mathbb{S} \simeq \Sigma^n\mathbb{S} \wedge \Sigma^m\mathbb{S} &\xrightarrow{f\wedge g} (E \wedge X) \wedge (E \wedge Y) \\ &\simeq (E \wedge E) \wedge (X \wedge Y) \xrightarrow{\mu\wedge \mathrm{id}} E \wedge (X \wedge Y) \end{aligned}$$

If E is a commutative ring spectrum, $\wedge$ is graded commutative in the sense that, for $x \in \widetilde{E}_n X$ and $y \in \widetilde{E}_m Y$, we have $\tau_*(x \wedge y) = (-1)^{nm} y \wedge x$, where $\tau\colon X \wedge Y \to Y \wedge X$ is the map switching the two factors.

2. *From the previous statement, we obtain, for $X = \mathrm{S}^0$, a product*

$$\widetilde{E}_n(\mathrm{S}^0) \otimes \widetilde{E}_m(\mathrm{S}^0) \longrightarrow \widetilde{E}_{n+m}(\mathrm{S}^0 \wedge \mathrm{S}^0) \cong \widetilde{E}_{n+m}(\mathrm{S}^0)$$

This multiplication has a unit element in $\widetilde{E}_0(\mathrm{S}^0) = [\mathbb{S}, E \wedge \mathrm{S}^0]$ given by the identity $i\colon \mathbb{S} \to E$.

Since $\widetilde{E}_n(\mathrm{S}^0) = [\Sigma^n\mathbb{S}, E \wedge \mathrm{S}^0] \cong \pi_n E$, this makes $\pi_* E$ into a unital graded ring, which is graded commutative if E is a commutative ring spectrum.

3. *We also obtain a natural associative bilinear product*

$$\wedge\colon \widetilde{E}^n X \otimes \widetilde{E}^m Y \longrightarrow \widetilde{E}^{n+m}(X \wedge Y)$$

by setting, for $f\colon \Sigma^\infty X \to \Sigma^n E$ and $g\colon \Sigma^\infty Y \to \Sigma^m E$,

$$\begin{aligned} f \wedge g\colon \Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty X \wedge \Sigma^\infty Y &\xrightarrow{f\wedge g} \Sigma^n E \wedge \Sigma^m E \\ &\simeq \Sigma^{n+m}(E \wedge E) \xrightarrow{\Sigma^{n+m}\mu} \Sigma^{n+m} E \end{aligned}$$

From this we also get an internal multiplication

$$\cup\colon \widetilde{E}^n X \otimes \widetilde{E}^m X \to \widetilde{E}^{n+m} X$$

by composing $\wedge$ with the map Δ^*, induced by the diagonal $\Delta\colon X \to X \wedge X$.

If E is a commutative ring spectrum, $\wedge$ and $\cup$ are graded commutative (for $\wedge$ this is meant in the same sense as for homology). Hence, in this case, $\widetilde{E}^* X$ becomes a non-unital graded ring which is graded commutative.

4. *In addition, we can give $\widetilde{E}^*X$ the structure of a graded left $\pi_{-*}E$-module by considering $\pi_{-n}E = [\Sigma^{-n}\mathbb{S}, E] \cong [\Sigma^\infty \mathrm{S}^0, \Sigma^n E] = \widetilde{E}^n(\mathrm{S}^0)$ and using the external multiplication $\wedge$ from the previous item.*

   By definition, this is compatible with the multiplication on $\widetilde{E}^*X$, in the sense that

$$(s \wedge x) \cup y = s \wedge (x \cup y) = (-1)^{nm} x \cup (s \wedge y)$$

   for $s \in \pi_{-n}E$, $x \in \widetilde{E}^m X$ and $y \in \widetilde{E}^* X$, where the second equality only holds if E is a ring spectrum.

All of these products are natural with respect to maps of (commutative) ring spectra.

These statements can be found in [Sw, p. 270ff.].

3.6 The K-Theory spectrum

Definition 3.34. Denote by $\mathrm{U}(n)$ the unitary group of degree n. Set $\mathrm{U} = \operatorname{colim}_n \mathrm{U}(n)$ where the colimit is taken over the inclusions $\iota_n \colon \mathrm{U}(n) \to \mathrm{U}(n+1)$, given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

In the following, the basepoint of $\mathrm{BU} \times \mathbb{Z}$ is some 0-cell in the component $\mathrm{BU} \times \{0\}$.

Proposition 3.35. *There are homotopy equivalences $\Omega(\mathrm{BU} \times \mathbb{Z}) \simeq \mathrm{U}$ and $\Omega\mathrm{U} \simeq \mathrm{BU} \times \mathbb{Z}$.*

This is [AGP, Corollary B.2.6 and Theorem B.2.7].

Definition 3.36. Denote by KU the Ω-spectrum given by $\mathrm{KU}_n = \mathrm{BU} \times \mathbb{Z}$ for n even, $\mathrm{KU}_n = \mathrm{U}$ for n odd and with structure maps the adjuncts of the homotopy equivalences from the previous proposition (to be more precise, we first replace it by a CW-spectrum; see Proposition 3.7).

Proposition 3.37. *For finite CW-complexes the spectrum* KU *represents reduced K-theory $\widetilde{\mathrm{K}}^*$ from Section 2.*

Furthermore KU *has the structure of a commutative ring spectrum such that its multiplication induces the product $\star$ on $\widetilde{\mathrm{K}}$.*

The first part can be found in [Sw, p. 216], the second part in [Sw, p. 300ff.]. Since KU is an Ω-spectrum, we have, for finite pointed CW-complexes X, a natural bijection $\varphi \colon [X, \mathrm{BU} \times \mathbb{Z}] \xrightarrow{\cong} \widetilde{\mathrm{K}}(X)$. We will now describe it. The idea is that BU classifies vector bundles up to stable isomorphism and $\mathbb{Z}$ classifies their virtual dimension. Let f be a representative of an element of $[X, \mathrm{BU} \times \mathbb{Z}]$.

First, to see what is happening, assume X to be connected and thus that its image lies in $\mathrm{BU} \times \{0\}$. Since X is finite, hence compact, its image lies only in finitely many cells. Thus there exists $n \in \mathbb{N}_0$ such that X lands in $\mathrm{BU}(n) \times \{0\} \subset \mathrm{BU} \times \{0\}$. Over this $\mathrm{BU}(n)$ we have the canonical n-dimensional vector bundle γ_n. Pulling it back, we obtain a vector bundle $f^*\gamma_n$ over X. Now $\varphi(f)$ is the equivalence class of $f^*\gamma_n$ in $\widetilde{\mathrm{K}}(X)$.

In the case that X is not connected, we need to remember that $\widetilde{\mathrm{K}}(X) \cong \ker(\mathrm{K}(X) \to \mathbb{Z})$, where $\mathrm{K}(X) \to \mathbb{Z}$ is given by the virtual dimension over the connected component of the basepoint. The image of f still lies in some $\mathrm{BU}(n) \times \mathbb{Z}$, so we can consider the pullback bundle $E = f^*(\gamma_n \times \mathbb{Z})$. For each connected component of $X_i \subset X$, we then subtract from E the bundle F_i which is trivial with dimension $\mathrm{pr}_{\mathbb{Z}}(f(X_i))$ over X_i and trivial with dimension 0 over $X \setminus X_i$. Hence we obtain a bundle over X, which has virtual dimension k over a connected component if its image under f is contained in $\mathrm{BU} \times \{k\}$. In particular it has virtual dimension 0 over the connected component of the basepoint. Thus it lies in $\widetilde{\mathrm{K}}(X)$ and it makes sense to say that $\varphi(f)$ is that bundle.

Note that, in the case that X is connected, the latter construction gives us the first one.

3.7 Associated cohomology theories of H-spaces

Definition 3.38. A *homotopy associative H-space* is a monoid in $(\mathbf{hCW}, \times)$. A homotopy associative H-space is *homotopy commutative* if it is a commutative monoid.

For Y a homotopy commutative and homotopy associative H-space, $\Sigma^\infty Y_+$ becomes a commutative ring spectrum via

$$\mu\colon \Sigma^\infty Y_+ \wedge \Sigma^\infty Y_+ \simeq \Sigma^\infty(Y_+ \wedge Y_+) \cong \Sigma^\infty(Y \times Y)_+ \xrightarrow{m_+} \Sigma^\infty Y_+$$

$$i\colon \mathbb{S} = \Sigma^\infty \mathrm{S}^0 \xrightarrow{\Sigma^\infty(\mathrm{const}_e)_+} \Sigma^\infty Y_+$$

where we consider $\mathrm{S}^0 = \mathrm{pt}_+$. We use that, by definition of the smash product, there is a homeomorphism $Y_+ \wedge Y_+ \cong (Y \times Y)_+$. This is the strong monoidalness of $\Sigma^\infty \circ (-)_+$ written out.

Hence $(\widetilde{\Sigma^\infty Y_+})^* X$ is a non-unital graded ring which is graded commutative. It is also, in a compatible way, a module over $\pi_*(\Sigma^\infty Y_+) = \pi_*^{\mathrm{s}}(Y_+)$.

Proposition 3.39. *Let $\mathbb{C}\mathrm{P}^\infty$ be the infinite complex projective space, i.e. the space $\mathrm{BU}(1)$ classifying complex line bundles. Also let γ be the universal line bundle over $\mathbb{C}\mathrm{P}^\infty$. Then the map $m\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ classifying the tensor product $\mathrm{pr}_1^*(\gamma) \otimes \mathrm{pr}_2^*(\gamma)$ together with an arbitrary element $e \in \mathbb{C}\mathrm{P}^\infty$ make $\mathbb{C}\mathrm{P}^\infty$ into a commutative H-space.*

Proof. We need to show that the diagrams from Definition 3.38 commute up to homotopy. We will derive this from the corresponding properties of the tensor product of vector bundles.

Since $(\mathrm{const}_e, \mathrm{id})^*(\mathrm{pr}_1^*(\gamma) \otimes \mathrm{pr}_2^*(\gamma)) \cong \underline{\mathbb{C}}^1 \otimes \gamma$, the map $m \circ (\mathrm{const}_e, \mathrm{id})$ classifies γ and hence is homotopic to the identity (as the identity also classifies γ). The analog holds for $(\mathrm{id}, \mathrm{const}_e)$.

Similarly, we have that the tensor product is associative and hence that the maps $m \circ (\mathrm{id} \times m)$ and $m \circ (m \times \mathrm{id})$ classify isomorphic bundles. Thus they are homotopic.

Again, in the same vein, $\tau^*(\mathrm{pr}_1^*(\gamma) \otimes \mathrm{pr}_2^*(\gamma)) \cong \mathrm{pr}_1^*(\gamma) \otimes \mathrm{pr}_2^*(\gamma)$ and hence $m \circ \tau \simeq m$. $\square$

This makes $(\widetilde{\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+})^*(X)$ into a non-unital graded ring and a $\pi_*^{\mathrm{s}}(\mathbb{C}\mathrm{P}^\infty_+)$-module.

4 Snaith's Theorem

In this section we describe an alternative way to obtain K-theory, based on $\mathbb{C}\mathrm{P}^\infty$. Since $\mathbb{C}\mathrm{P}^\infty$ classifies complex line bundles, we could also say that, in some sense, we will obtain K-theory from those.

We first need to collect a variety of well-known results from algebraic topology that we will use. Subsequently we will formulate Snaith's theorem and then prove (parts of) it, using as main input two results, one of Snaith himself [Sn] and one of Segal [Se].

4.1 Collection of general tools

The following is a powerful tool for computing generalized homology groups (there is also a version for cohomology but we will not need it). We will use it to compute rational stable homotopy groups.

Proposition 4.1 (Atiyah–Hirzebruch Spectral Sequence)**.** *Let $\tilde{h}_*$ be a generalized reduced homology theory such that $\tilde{h}_n(\mathrm{S}^0)$ is bounded below (i.e. there exists $n \in \mathbb{Z}$ such that $\tilde{h}_k(\mathrm{S}^0) = 0$ for $k \leq n$). Also let X be a CW-complex. If $\tilde{h}_*$ is additive or X is finite, then we have a natural homology spectral sequence*

$$E^2_{p,q} = \mathrm{H}_p(X; \tilde{h}_q(\mathrm{S}^0)) \Rightarrow \tilde{h}_{p+q}(X_+)$$

This is proven in [Ko, Theorem 4.2.5].

The next statement is the analog of the Künneth and Universal Coefficient Theorems for the left exact functor lim.

Proposition 4.2 (Milnor exact sequence)**.** *Let $\tilde{h}^*$ be an additive generalized reduced cohomology theory, X a CW-complex, and $(X_n)_{n \in \mathbb{N}_0}$ an ascending sequence of subcomplexes such that their union is all of X. Then, for any $k \in \mathbb{Z}$, there is a short exact sequence of abelian groups*

$$0 \longrightarrow \lim_n{}^1 \tilde{h}^{k-1}(X_n) \longrightarrow \tilde{h}^k(X) \longrightarrow \lim_n \tilde{h}^k(X_n) \longrightarrow 0$$

where the third map is induced by the inclusions $X_n \to X$.

This can be found in [HaAT, Theorem 3F.8]. The only fact we will need about $\lim^1$ is that it is zero if all of the groups it has as input are zero.

Corollary 4.3. *Let X be a CW-complex and $(X_n)_{n\in\mathbb{N}_0}$ an ascending sequence of finite subcomplexes such that their union is all of X. If X has only cells of even dimension, then the natural map*

$$\widetilde{\mathrm{KU}}^0(X) \longrightarrow \lim_n \widetilde{\mathrm{KU}}^0(X_n)$$

given by the inclusions $X_n \to X$, is an isomorphism of groups.

Proof. Consider the Milnor exact sequence

$$0 \longrightarrow \lim_n{}^1 \widetilde{\mathrm{KU}}^{-1}(X_n) \longrightarrow \widetilde{\mathrm{KU}}^0(X) \longrightarrow \lim_n \widetilde{\mathrm{KU}}^0(X_n) \longrightarrow 0$$

Since X_n is finite, we have $\widetilde{\mathrm{KU}}^{-1}(X_n) \cong \widetilde{\mathrm{K}}^{-1}(X_n)$ which is trivial by a computation we did in Section 2, see Lemma 2.44. □

Definition 4.4. Write MU for the complex Thom spectrum.

We will not actually need to know how MU is defined, only some statements about it. One of them is the following classical result.

Proposition 4.5. *We have $\pi_*(\mathrm{MU}) \cong \mathbb{Z}[x_1, x_2, \dots]$ with x_i living in degree $2i$.*

This can be found in [Ra, Theorem 3.1.5].

The last thing in this subsection is a statement about generalized cohomology, roughly corresponding to the uniqueness of singular cohomology on CW-complexes.

Proposition 4.6. *Let $\tilde{h}^*$ and $\tilde{g}^*$ be additive generalized reduced cohomology theories and $\eta\colon \tilde{h}^* \Rightarrow \tilde{g}^*$ a map. If $\eta_{\mathrm{S}^0}\colon \tilde{h}^*(\mathrm{S}^0) \to \tilde{g}^*(\mathrm{S}^0)$ is an isomorphism, then $\eta_X\colon \tilde{h}^*X \to \tilde{g}^*X$ is an isomorphism for all finite dimensional pointed CW-complexes X.*

This is a special case of [Sw, Theorem 7.67].

4.2 Statement of the theorem

Consider the pointed map $\phi\colon \mathbb{C}\mathrm{P}^\infty_+ \to \mathrm{BU} \times \mathbb{Z}$ given by the inclusion $\mathbb{C}\mathrm{P}^\infty \simeq \mathrm{BU}(1) \to \mathrm{BU} \times \{1\}$ and sending $+$ to the basepoint $(*, 0)$. We know that KU is a Ω-spectrum. Hence, by Proposition 3.31, $\Omega^\infty\mathrm{KU} \simeq \mathrm{BU} \times \mathbb{Z}$ and we obtain, from ϕ, a map $\Phi\colon \Sigma^\infty\mathbb{C}\mathrm{P}^\infty_+ \to \mathrm{KU}$. We want to see that it is a map of (commutative) ring spectra.

We can describe the classifying space BU as the colimit over the complex Grassmannians $(\mathrm{Gr}_{n,k})_{n\leq k}$ with maps the inclusions

$$\begin{aligned} i_{n,k}\colon \mathrm{Gr}_{n,k} &\longrightarrow \mathrm{Gr}_{n,k+1}, \qquad W \longmapsto W \oplus 0 \\ j_{n,k}\colon \mathrm{Gr}_{n,k} &\longrightarrow \mathrm{Gr}_{n+1,k+1}, \quad W \longmapsto \mathbb{C} \oplus W \end{aligned}$$

(see e.g. [AGP, p. 438f.]). In particular we have $\mathrm{BU} \cong \mathrm{colim}_n\, \mathrm{Gr}_{n,2n-1}$, where the colimit is taken over the maps

$$\iota_n\colon i_{n+1,2n} \circ j_{n,2n-1}\colon \mathrm{Gr}_{n,2n-1} \to \mathrm{Gr}_{n+1,2n+1}$$

since this is cofinal in the larger diagram. Also note that $\mathbb{C}\mathrm{P}^\infty \cong \operatorname{colim}_n \mathrm{Gr}_{1,n}$ taken over the inclusions $i_{1,n}$.

Let the pointed map $\phi\colon \mathbb{C}\mathrm{P}^\infty_+ \to \mathrm{BU} \times \mathbb{Z}$ be given by the inclusion $\mathbb{C}\mathrm{P}^\infty \to \mathrm{BU} \times \{1\}$ and $+ \mapsto (*, 0)$. We also write, for $n \geq 1$,

$$G_n = \mathrm{Gr}_{n,2n-1} \times \{-n, -n+1, \ldots, n\}$$

and $\iota_n\colon G_n \to G_{n+1}$ given by $\iota'_n = \iota_n \times f_n$ where f_n is the canonical inclusion $\{-n, \ldots, n\} \to \{-(n+1), \ldots, n+1\}$. Note that $\phi((\mathrm{Gr}_{1,n})_+) \subset \mathrm{Gr}_{n,2n-1} \times \{0,1\} \subset G_n$. Thus we get a commutative diagram

$$\begin{array}{ccc}
[\mathbb{C}\mathrm{P}^\infty_+ \wedge \mathbb{C}\mathrm{P}^\infty_+, \mathbb{C}\mathrm{P}^\infty_+] & \longrightarrow & \lim_n [(\mathrm{Gr}_{1,n})_+ \wedge (\mathrm{Gr}_{1,n})_+, \mathbb{C}\mathrm{P}^\infty_+] \\
\Big\downarrow{\scriptstyle \phi_*} & & \Big\downarrow{\scriptstyle \phi_*} \\
[\mathbb{C}\mathrm{P}^\infty_+ \wedge \mathbb{C}\mathrm{P}^\infty_+, \mathrm{BU} \times \mathbb{Z}] & \xrightarrow{\cong} & \lim_n [(\mathrm{Gr}_{1,n})_+ \wedge (\mathrm{Gr}_{1,n})_+, \mathrm{BU} \times \mathbb{Z}] \\
\Big\uparrow{\scriptstyle (\phi\wedge\phi)^*} & & \Big\uparrow{\scriptstyle (\phi\wedge\phi)^*} \\
[(\mathrm{BU} \times \mathbb{Z}) \wedge (\mathrm{BU} \times \mathbb{Z}), \mathrm{BU} \times \mathbb{Z}] & \xrightarrow{\cong} & \lim_n [G_n \wedge G_n, \mathrm{BU} \times \mathbb{Z}]
\end{array}$$

where the limits are taken over the maps $(i_{1,n})_+ \wedge (i_{1,n})_+$ respectively ι'_n. Note that here the uppermost two objects are only (pointed) sets, i.e. we do not consider them to have any group structure. The lower two horizontal maps are isomorphisms by Corollary 4.3. For this we use that $\mathrm{Gr}_{n,k}$, and hence $\mathrm{Gr}_{n,k} \wedge \mathrm{Gr}_{n,k}$, has a CW-complex structure with only cells of even dimension. (For a cell structure on the real Grassmannians see e.g. the classical [MS, §6]. The complex case works the same in principle; it can be found in [GH, Chapter 1, 5].)

Denote by $\gamma_{n,k}$ the canonical n-dimensional bundle over $\mathrm{Gr}_{n,k}$. Set $v_{1,n}$ the virtual bundle over $(\mathrm{Gr}_{1,n})_+$ given by $\gamma_{1,n}$ over $\mathrm{Gr}_{1,n}$ and $\underline{\mathbb{C}}^0$ over $+$, and u_n the virtual bundle over G_n given by $\gamma_{n,2n-1} + \underline{\mathbb{C}}^{k-n}$ over $\mathrm{Gr}_{n,2n-1} \times \{k\}$. Note that $\iota_n^*(u_{n+1}) \cong u_n$, $i_{1,n}^*(v_{1,n+1}) \cong v_{1,n}$, and $\phi|_{(\mathrm{Gr}_{1,n})_+}(u_{n,2n-1}) \cong v_{1,n}$.

Everything we will do in the next paragraph can be summarized in the following diagram. We state it now, so that one may refer to it when reading the subsequent text.

$$\begin{array}{ccc}
v'_1 \wedge v'_1 & \longmapsto & (v'_{1,n} \wedge v'_{1,n})_n \\
\Big\downarrow & & \Big\downarrow \\
v_1 \wedge v_1 & \longmapsto & (v_{1,n} \wedge v_{1,n})_n \\
\Big\uparrow & & \Big\uparrow \\
u \wedge u & \longmapsto & (u_n \wedge u_n)_n
\end{array}$$

Let $u_n \wedge u_n = \mathrm{pr}_1^*(u_n) \cdot \mathrm{pr}_2^*(u_n)$, where pr_1 and pr_2 are the two projections $G_n \wedge G_n \to G_n$. Analogously define the bundle $v_{1,n} \wedge v_{1,n}$ over $(\mathrm{Gr}_{1,n})_+ \wedge (\mathrm{Gr}_{1,n})_+$. These are compatible, hence they give us elements $(u_n \wedge u_n)_n$ respectively $(v_{1,n} \wedge v_{1,n})_n$ of the two lower limit-terms on the right hand side (here we identify a bundle with the homotopy class of maps classifying it). Since $v_{1,n} \wedge v_{1,n}$ is, on $\mathbb{CP}^\infty$, a virtual (actually genuine) line bundle, its classifying map already lands in $\phi(\mathbb{CP}^\infty_+) \subset \mathrm{BU} \times \{0,1\}$ (up to homotopy). Hence we can lift $v_{1,n} \wedge v_{1,n}$ along ϕ_* to an element $v'_{1,n} \wedge v'_{1,n} \in [(\mathrm{Gr}_{1,n})_+ \wedge (\mathrm{Gr}_{1,n})_+, \mathbb{CP}^\infty_+]$. These are again compatible, hence they give us an element $(v'_{1,n} \wedge v'_{1,n})_n$ of the limit term in the upper right. These bundles also are, by definition, the pullback under the inclusion $(\mathrm{Gr}_{1,n})_+ \to \mathbb{CP}^\infty_+$ of the bundle $v'_1 \wedge v'_1 = \mathrm{pr}_1^*(v_1) \otimes \mathrm{pr}_2^*(v_1)$ over $\mathbb{CP}^\infty_+$, where v_1 is the bundle over $\mathbb{CP}^\infty_+$ given by the canonical bundle γ_1 over $\mathbb{CP}^\infty$ and $\underline{\mathbb{C}}^0$ over $+$. Denote by $v_1 \wedge v_1 = \phi_*(v'_1 \wedge v'_1)$, i.e. the virtual bundle corresponding to $v'_1 \wedge v'_1$. By commutativity of the diagram and injectivity of the middle map, we get that $(\phi \wedge \phi)^*(u \wedge u) = v_1 \wedge v_1$.

We write m' for the map $u \wedge u\colon (\mathrm{BU} \times \mathbb{Z}) \wedge (\mathrm{BU} \times \mathbb{Z}) \to \mathrm{BU} \times \mathbb{Z}$ and denote by m the H-space multiplication on $\mathbb{CP}^\infty$. Consider the following diagram

$$\begin{array}{ccc} \mathbb{CP}^\infty_+ \wedge \mathbb{CP}^\infty_+ & \xrightarrow{m_+} & \mathbb{CP}^\infty_+ \\ {\scriptstyle \phi\wedge\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ (\mathrm{BU} \times \mathbb{Z}) \wedge (\mathrm{BU} \times \mathbb{Z}) & \xrightarrow{m'} & \mathrm{BU} \times \mathbb{Z} \end{array}$$

We want to see that it commutes up to homotopy. Our previous considerations tell us that

$$(\phi \wedge \phi)^*(u \wedge u) = v_1 \wedge v_1 = \phi_*(v'_1 \wedge v'_1)$$

This is exactly the statement that $m' \circ (\phi \wedge \phi) \simeq \phi \circ m_+$, since, by definition of $v'_1 \wedge v'_1$, it is homotopic to m. Hence ϕ is a map of monoids in $\mathbf{hCW}_*$.

By Proposition 3.29, our adjunction $\Sigma^\infty \dashv \Omega^\infty$ lifts to an adjunction between $\mathbf{Mon}_{\mathbf{hCW}_*}$ and $\mathbf{Mon}_{\mathbf{HoSpec}}$. Thus, to see that the map Φ is a map of ring spectra, it would be enough to prove that the monoid structure on $\Omega^\infty\mathrm{KU} \simeq \mathrm{BU} \times \mathbb{Z}$ coming from the structure of a monoidal functor on Ω^∞ (which is induced by Σ^∞) is the same as the one we constructed above.

This seems like it should be true since in the zeroth component the multiplication $\mathrm{KU} \wedge \mathrm{KU} \to \mathrm{KU}$ is just given by $u \wedge u\colon (\mathrm{BU} \times \mathbb{Z}) \wedge (\mathrm{BU} \times \mathbb{Z}) \to \mathrm{BU} \times \mathbb{Z}$ (cf. the construction in [Sw, p. 300ff.] we already referred to when introducing KU). Sadly this seems to be cumbersome to prove in the framework we have presented here since the action of Ω^∞ on maps is not given in a very explicit manner, at least in the literature (the problem is that this is already true for the so called spectrification which is a process that replaces a CW-spectrum by a homotopy equivalent Ω-spectrum and which lies at the heart of Ω^∞). Hence we have to refer to alternative settings of stable homotopy theory where this statement follows more easily; see e.g. [nL].

Let $\iota\colon \mathrm{S}^2 \cong \mathbb{CP}^1 \to \mathbb{CP}^\infty$ be the inclusion. Consider the composition

$$\Sigma^2\mathbb{S} \xrightarrow{\cong} \Sigma^\infty\mathrm{S}^2 \longrightarrow \Sigma^\infty\mathrm{S}^2 \vee \mathbb{S} \xrightarrow{\simeq} \Sigma^\infty\mathrm{S}^2_+ \xrightarrow{\Sigma^\infty\iota_+} \Sigma^\infty\mathbb{CP}^\infty_+$$

where we use that $\Sigma^\infty\mathrm{S}^2$ is cofinal in $\Sigma^2\mathbb{S}$ for the isomorphism and Lemma 3.19 for the homotopy equivalence. This gives us an element $\beta \in \pi_2^{\mathrm{s}}(\mathbb{CP}^\infty_+)$.

As we said before, $\widetilde{(\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+)}^*(X)$ is a ring and a module over $\pi^{\mathrm{s}}_{-*}(\mathbb{C}\mathrm{P}^\infty_+)$ with compatible multiplications. We want to localize it at β to get a ring $\widetilde{(\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+)}^*(X)[\beta^{-1}]$. Sadly neither of the first two rings is commutative, only graded commutative. Luckily there is still a theory of localization in the non-commutative case, as long as the multiplicative set we are localizing at fulfills the so called Ore condition [Sk]. Since $\beta \in \pi^{\mathrm{s}}_2(\mathbb{C}\mathrm{P}^\infty_+)$ lies in even degree, it commutes with everything, and the multiplicative set $B = \{\beta^n \mid n \in \mathbb{N}_0\}$ fulfills the Ore condition. Using the discussion in [Sk, Section 7], one checks, as in the commutative case, that $\widetilde{(\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+)}^*(X)[\beta^{-1}]$ is again a ring and that this localization is exact.

We now want to obtain a map

$$\Psi_X \colon \widetilde{(\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+)}^*(X)[\beta^{-1}] \longrightarrow \widetilde{\mathrm{KU}}^*(X)$$

Recall that H denotes the canonical line bundle over $\mathbb{C}\mathrm{P}^1 \cong \mathrm{S}^2$. By Bott periodicity, we have that multiplication with the element

$$H - 1 \in \widetilde{\mathrm{K}}(\mathrm{S}^2) \cong \widetilde{\mathrm{KU}}^0(\mathrm{S}^2) \cong \pi_2(\mathrm{KU})$$

is invertible (since it has an inverse in $\pi_{-2}(\mathrm{KU})$). Hence it is enough to prove that Φ sends multiplication with β to multiplication with $H-1$, i.e. that $[\Phi \circ \beta] = H - 1 \in \widetilde{\mathrm{KU}}^0(\mathrm{S}^2)$. By the naturality of the adjunction bijection, for $\Sigma^\infty \dashv \Omega^\infty$, we obtain that $\Phi \circ \Sigma^\infty \iota_+ = \phi \circ \iota_+$, which is the map classifying the bundle $v_{1,2}$ over S^2_+ given by H over S^2 and $\underline{\mathbb{C}}^0$ over $+$. Hence it only remains to see that the map $\widetilde{\mathrm{KU}}^0(\mathrm{S}^2_+) \to \widetilde{\mathrm{KU}}^0(\mathrm{S}^2)$ induced by

$$\Sigma^\infty \mathrm{S}^2 \longrightarrow \Sigma^\infty \mathrm{S}^2 \vee \mathbb{S} \xrightarrow{\simeq} \Sigma^\infty \mathrm{S}^2_+$$

maps $v_{1,2}$ to $H - 1$.

For this note that, as $\vee$ constitutes the coproduct of **HoSpec**, $\Sigma^\infty \mathrm{S}^2_+ \simeq \Sigma^\infty \mathrm{S}^2 \vee \mathbb{S}$ induces a splitting $\widetilde{\mathrm{KU}}^0(\mathrm{S}^2_+) \cong \widetilde{\mathrm{KU}}^0(\mathrm{S}^2) \oplus \widetilde{\mathrm{KU}}^0(\mathrm{S}^0)$. This is equivalent to the splitting $\widetilde{\mathrm{K}}(\mathrm{S}^2_+) \cong \widetilde{\mathrm{K}}(\mathrm{S}^2) \oplus \widetilde{\mathrm{K}}(\mathrm{S}^0)$ from Remark 2.20 since the map $\widetilde{\mathrm{KU}}^0(\mathrm{S}^0) \to \widetilde{\mathrm{KU}}^0(\mathrm{S}^2_+)$ induced by

$$\Sigma^\infty \mathrm{S}^2_+ \xrightarrow{\simeq} \Sigma^\infty \mathrm{S}^2 \vee \mathbb{S} \longrightarrow \mathbb{S}$$

is homotopic to the one induced by the map $\mathrm{S}^2_+ \to \mathrm{S}^0$ collapsing S^2 to a point (this follows by inspection of the map in Lemma 3.19), which corresponds to the split $\widetilde{\mathrm{K}}(\mathrm{S}^0) \to \widetilde{\mathrm{K}}(\mathrm{S}^2_+)$. Thus the map $\widetilde{\mathrm{KU}}^0(\mathrm{S}^2_+) \to \widetilde{\mathrm{KU}}^0(\mathrm{S}^2)$ is equivalent to the other split $\widetilde{\mathrm{K}}(\mathrm{S}^2_+) \to \widetilde{\mathrm{K}}(\mathrm{S}^2)$. Hence it also maps $v_{1,2}$ to $H - 1$.

We have now shown that the map Ψ_X exists as claimed. This means we can now state the theorem this section is about.

Theorem 4.7 (Snaith's Theorem)**.** *Let X be a finite dimensional pointed CW-complex. Then the natural map*

$$\Psi_X \colon \widetilde{(\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+)}^*(X)[\beta^{-1}] \longrightarrow \widetilde{\mathrm{KU}}^*(X)$$

from above is an isomorphism of rings.

4.3 The proof

We first state the two results constituting the main input.

Proposition 4.8. *Let* $b_U \in \pi_2^{\mathrm{s}}(\mathrm{BU}_+)$ *be the composition*

$$\Sigma^2\mathbb{S} \xrightarrow{\cong} \Sigma^\infty \mathrm{S}^2 \longrightarrow \Sigma^\infty \mathrm{S}^2 \vee \mathbb{S} \xrightarrow{\simeq} \Sigma^\infty \mathrm{S}^2_+ \longrightarrow \Sigma^\infty \mathrm{BU}_+$$

Then there is an isomorphism of rings

$$(\widetilde{\Sigma^\infty \mathrm{BU}_+})^0(X)[b_U^{-1}] \cong \operatorname*{colim}_n \prod_{-n \leq k} \widetilde{\mathrm{MU}}^{2k}(X)$$

for all finite dimensional pointed CW-complexes X*.*

This is the ingredient we will use from Snaith's original paper [Sn, Theorem 2.7].

Proposition 4.9. *Let* X *be a CW-complex. Then the map*

$$\Phi^*\colon (\widetilde{\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+})^0(X) \longrightarrow \widetilde{\mathrm{KU}}^0(X)$$

is surjective.

This is proven in [Se].

We need two more lemmas, then we will finally by able to prove Snaith's Theorem.

Lemma 4.10. *There is no torsion in* $\pi_*^{\mathrm{s}}(\mathbb{C}\mathrm{P}^\infty_+)[\beta^{-1}]$*.*

Proof. Denote by $\det_n\colon \mathrm{U}(n) \to \mathrm{U}(1)$ the determinant. These maps are compatible with the inclusions $\iota_n\colon \mathrm{U}(n) \to \mathrm{U}(n+1)$ from Definition 3.34 and hence combine to a map $\det\colon \mathrm{U} \to \mathrm{U}(1)$. Also writing $i\colon \mathrm{U}(1) \to \mathrm{U}$ for the inclusion, we have $\det \circ i = \mathrm{id}$. Hence, after applying the functor $\pi_*^{\mathrm{s}} \circ (-)_+ \circ \mathrm{B}$, we obtain a split injective map

$$\pi_*^{\mathrm{s}}((\mathrm{B}i)_+)\colon \pi_*^{\mathrm{s}}(\mathbb{C}\mathrm{P}^\infty_+) \longrightarrow \pi_*^{\mathrm{s}}(\mathrm{BU}_+)$$

Since $\mathrm{B}i\colon \mathbb{C}\mathrm{P}^\infty \to \mathrm{BU}$ is just the canonical inclusion, the map $\pi_*^{\mathrm{s}}((\mathrm{B}i)_+)$ sends β to b_U. Hence we obtain a map

$$\pi_*^{\mathrm{s}}(\mathbb{C}\mathrm{P}^\infty_+)[\beta^{-1}] \longrightarrow \pi_*^{\mathrm{s}}(\mathrm{BU}_+)[b_U^{-1}]$$

which is again injective. Since, by Proposition 4.5 and Proposition 4.8, the ring $\pi_*^{\mathrm{s}}(\mathrm{BU}_+)[b_U^{-1}]$ is torsion free, so is $\pi_*^{\mathrm{s}}(\mathbb{C}\mathrm{P}^\infty_+)[\beta^{-1}]$. ☐

Lemma 4.11. *We have*

$$\pi_k^{\mathrm{s}}(\mathbb{C}\mathrm{P}^\infty_+) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & k \geq 0 \textit{ even} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Since the stable homotopy groups $\pi_*^{\mathrm{s}} = \widetilde{\mathbb{S}}_*$ form an additive generalized reduced homology theory and $\mathbb{Q}$ is flat, the rational stable homology groups $\pi_*^{\mathrm{s}} \otimes \mathbb{Q}$ also form an additive generalized homology theory. By Remark 3.17, we have $\pi_n^{\mathrm{s}} X = \pi_n(\Sigma^\infty X) \cong \operatorname{colim}_k \pi_{n+k}(\Sigma^k X)$. Hence, for $n < 0$, we have $\pi_n^{\mathrm{s}}(\mathrm{S}^0) = 0$ and thus also $\pi_n^{\mathrm{s}}(\mathrm{S}^0) \otimes \mathbb{Q} = 0$. It follows that we can use the Atiyah–Hirzebruch spectral sequence (Proposition 4.1) for $\tilde{h}_* = \pi_*^{\mathrm{s}} \otimes \mathbb{Q}$ and $X = \mathbb{C}\mathrm{P}_+^\infty$.

We will need to know the rational stable homotopy groups of S^0. By a theorem of Serre, the homotopy groups $\pi_n(\mathrm{S}^k)$ are finite except if $n = k$ or k even and $n = 2k - 1$ [Hu, XI, Theorem 7.1 and Theorem 12.1]. Thus, for $n > 0$, $\pi_n^{\mathrm{s}}(\mathrm{S}^0)$ is finite and $\pi_n^{\mathrm{s}}(\mathrm{S}^0) \otimes \mathbb{Q} = 0$. Since $\pi_n(\mathrm{S}^n) = \mathbb{Z}$ and $\pi_n(\mathrm{S}^n) \to \pi_{n+1}(\mathrm{S}^{n+1})$ is an isomorphism, we have $\pi_0^{\mathrm{s}}(\mathrm{S}^0) = \mathbb{Z}$ and $\pi_0^{\mathrm{s}}(\mathrm{S}^0) \otimes \mathbb{Q} = \mathbb{Q}$.

We get that

$$E^2_{p,q} = \mathrm{H}_p(\mathbb{C}\mathrm{P}^\infty; \pi_q^{\mathrm{s}}(\mathrm{S}^0) \otimes \mathbb{Q}) \cong \begin{cases} \mathrm{H}_p(\mathbb{C}\mathrm{P}^\infty; \mathbb{Q}) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Hence the spectral sequence already collapses at the second page and there are no extension problems. We obtain $\pi_n^{\mathrm{s}}(\mathbb{C}\mathrm{P}_+^\infty) \cong \mathrm{H}_n(\mathbb{C}\mathrm{P}^\infty; \mathbb{Q})$, which is the desired result. □

We now put everything together.

Proof of Snaith's Theorem. Since localization is exact, we have that the domain is a cohomology theory. Hence, by Proposition 4.6, we only need to prove that

$$\Psi^n_{\mathrm{S}^0} \colon (\widetilde{\Sigma^\infty \mathbb{C}\mathrm{P}_+^\infty})^n(\mathrm{S}^0)[\beta^{-1}] \longrightarrow \widetilde{\mathrm{KU}}^n(\mathrm{S}^0)$$

is an isomorphism for all $n \in \mathbb{Z}$. As both sides are compatibly two periodic, it is actually even enough to show it for $\Psi^0_{\mathrm{S}^0}$ and $\Psi^{-1}_{\mathrm{S}^0}$ or equivalently for $\Psi^0_{\mathrm{S}^0}$ and $\Psi^0_{\mathrm{S}^1}$. By Lemma 4.10 and Lemma 4.11, we have that the domain of $\Psi^0_{\mathrm{S}^0}$ is isomorphic to $\mathbb{Z}$ and that the domain of $\Psi^0_{\mathrm{S}^1}$ is trivial. The same holds for the respective targets. Since, by Proposition 4.9, both maps are surjective, they are already isomorphisms. □

References

[Ad] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago – London, 1974.

[AGP] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic topology from a homotopical viewpoint.* Universitext. Springer-Verlag, New York, 2002.

[At] M. F. Atiyah. *K-theory.* Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[Bre] Glen E. Bredon. *Topology and geometry*. Vol. 139. Graduate Texts in Mathematics. Springer-Verlag, New York, 1993.

[Bro] Ronald Brown. *Topology and groupoids*. Printed and distributed by Createspace, 2006. URL: `https://groupoids.org.uk/topgpds.html`.

[GH] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978.

[HaAT] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. URL: `https://www.math.cornell.edu/~hatcher/AT/ATpage.html`.

[HaK] Allen Hatcher. *Vector Bundles and K-Theory*. 2009. URL: `https://www.math.cornell.edu/~hatcher/VBKT/VBpage.html`.

[Hu] Sze-Tsen Hu. *Homotopy theory*. Vol. 8. Pure and Applied Mathematics. Academic Press, New York and London, 1959.

[Ke] G. M. Kelly. "Doctrinal adjunction". In: *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*. Ed. by Gregory M. Kelly. Springer Berlin Heidelberg, 1974, pp. 257–280.

[Ko] S. O. Kochman. *Bordism, stable homotopy and Adams spectral sequences*. Vol. 7. Fields Institute Monographs. American Mathematical Society, Providence, RI, 1996.

[Ma] Saunders Mac Lane. *Categories for the working mathematician*. Second edition. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[MS] John W. Milnor and James D. Stasheff. *Characteristic classes*. Vol. 76. Annals of Mathematics Studies. Princeton University Press, Princeton, N. J. and University of Tokyo Press, Tokyo, 1974.

[nL] nLab. *monoidal adjunction — nLab*. Retrieved June 26th, 2017. URL: `https://ncatlab.org/nlab/show/monoidal+adjunction`.

[Ra] Douglas C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. 2nd ed. AMS Chelsea Publishing, Providence, RI, 2004. URL: `https://web.math.rochester.edu/people/faculty/doug/mu.html`.

[Ru] Yuli B. Rudyak. *On Thom spectra, orientability, and cobordism*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[Se] Graeme Segal. "The stable homotopy of complex projective space". In: *Quart. J. Math. Oxford Ser. (2)* 24 (1973), pp. 1–5.

[Sk] Zoran Škoda. "Noncommutative localization in noncommutative geometry". In: *Non-commutative localization in algebra and topology*. Vol. 330. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2006, pp. 220–313. arXiv: `math/0403276`.

[Sn] Victor Snaith. "Localized stable homotopy of some classifying spaces". In: *Math. Proc. Cambridge Philos. Soc.* 89.2 (1981), pp. 325–330.

[Sw] Robert M. Switzer. *Algebraic topology — homology and homotopy.* Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2002.